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A. FINITE ELEMENTS FOR FLUID DYNAMICS

B. SUPERCRITICAL DESIGN

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The first part of this report (1) investigates a given overdetermined set of 1st order, partial differential equations and reduces them to the proper one for given initial and/or boundary conditions, in a natural representation for finite-difference descretizations and calculations. Alternative variational formulations are then developed and the same questions (of determinancy and well-posedness) asked with respect to finite-elements discretizations based on them. We then move (Part II) to the actual solution of the

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Tricomi problem, based on the variational formulations proposed and on finite elements descretizations. The problem involves a mixed-type (elliptic hyperbolic) domain, and serves as a model for transonic flows, when represented in the hodograph plane. The following 3 parts (III, IV, V) are related to a new design procedure, proposed for transonic airfoils and aircraft. The solution of a non-standard elliptic boundary-value problem is involved and its mathematical (III, IV) and numerical (V) aspects are investigated. The latter part, exposing and implementing the new procedure, is preliminary. It needs further development, yet may hold versatility and possibilities for beyond other design procedures currently available.

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REDUNDANT SYSTEM AND VARIATIONAL PRINCIPLES

IN CONTINUUM MECHANICS

Nima Geffen

Abstract

The mathematical modeling of a physical continuum by a set of
first-order partial differential equations is commonly redundant,
i.e., it includes more equations than unknowns. A simple analysis
gives the reduced set, which fully determines the field quantities
satisfying appropriately prescribed initial and boundary conditions.
An algebraic look at the system provides the appropriate choice
for descretization, which is a necessary condition for solvability
and stability.

The same is tried for a number of variational representations
of the field in terms of primitive variables. A few of the latter
formulations are new.

1.   Forward.

Two forms for mathematical modeling of physical continua are
described and discussed in two parts: I., a system of first order,
partial differential equations for the field components and quantities.
II., an integral representation of the fields via a variational prin-
ciple; a functional of the field quantities becomes stationary at the
solution-point of the differential system which constitutes it's
Euler-Lagrange conditions. The two approaches are common in physics
(e.g. the Newtonian and Lagrangian representations of mechanics,
Hamilton's principle of least action) and both have been extremely
useful, illuminating,and necessary in many and varied areas, both in
classical (e.g. mechanics, electrodynamics, relativity e.g. [1], [2],
[3])and quantum applications [4].

The common presentation (e.g. [1], [2], [3], [4]) in theoretical
physics ignors the question of well posedness. The principle of
least action in mechanics is formulated as a boundary value problem,
where the initial and final locations and times are prescribed
while in fact it is an initial value problem where location, time
and velocity are given initially and evolve according to the laws
of motion. The same is true for the standard variational representa-
tion of the electromagnetic field, relativity and quantum electro-
dynamic.

For analysis and simulation of a particular physical or engin-
eering situation,appropriate initial and boundary conditions have to
be considered, the problem has to be well posed, i.e. to be solvable

uniquely and stably, and the model has to admit geometries and non-smoothness of interest in practice.

Initial and boundary conditions are incorporated in the following analysis for both the differential and variational formulations. A few of the variational formulations, we believe, are new, as well as viewing the redundancy in their Euler equations as a possible cause for computational instabilities. The initial motivation for this work came from finite-difference [5] and finite-element [6] simulation of problems in transonic aerodynamics.

The question and system treated are elementary and so general, the analysis and answer so simple, to be judged trivial, if not for the fact that the question does come up occasionally and the answer not always immediate. Essentially the same problem, we beleive, has been treated recently (and has come to our attention while writing this note), in a completely different context for electromagnetic field equations ([7], [8]). It also seems to be related to other recent approaches [9] and to variational formulation and finite element analyses of analogous continuum systems of engineering importance [10],[11],[12].

2. General Equations.

Continuum mechanics, electrodynamics and other physical theories can be modelled by various specialization of the following set of first order, partial differential equations specifying the source distribution of the vector field $\underline{A}$:

$$(1) \qquad \nabla \cdot \underline{A} = G$$

and the curl of the vector field $\underline{u}$:

$$(2) \qquad \nabla \times \underline{u} = \underline{W}$$

for $n + 1$ independent variables:

(i) $\qquad \underline{x} = (x_o, x_1, \ldots, x_n)$

where x_0 may designate time, (or a time-like coordinate).

m dependent variables:

(ii) $\qquad \underline{u} = \underline{u}(\underline{x}) = (u_1, \ldots, u_m)(x_o, x_1, \ldots, x_n)$

and:

(iii) $\qquad \underline{A} = \underline{A}(\underline{x}, \underline{u}) = (A_0, \ldots, A_n)$

(iv) $\qquad G = G(\underline{x}, \underline{u})$

(v) $\qquad \underline{W} = \underline{W}(\underline{x}) = (W_1, \ldots, W_n).$

If $\underline{u}$ appears linearly in (iii), i.e.:

(vi) $\quad \dfrac{\partial \underline{A}}{\partial \underline{u}} = \dfrac{\partial A_i}{\partial u_j} = c_{ij}(\underline{x})$ where the $c_{ij}(\underline{x})$ do not depend on

$\underline{u}$, which, in addition, does not appear in (iv) i.e.:

$G = G(\underline{x})$ the system (1), (2) is linear.

If $G = G(\underline{x}, \underline{u})$ the system is semi-linear. If $\underline{u}$ appears

in $\underline{A}$ nonlinearly, i.e.:

(vii) $\quad \dfrac{\partial \underline{A}}{\partial \underline{u}} = \dfrac{\partial A_i}{\partial u_j} = f_{ij}(\underline{u}, \underline{x})$ the system is quasilinear.

The vorticity $\underline{W}$ has to satisfy a compatibility condition,

$$(2a) \qquad\qquad\qquad \nabla \cdot \underline{W} = 0$$

and not all vorticity distributions are admissible.

The system (1), (2) can be extended to coupled fields
$\underline{u}(x)$, $\underline{v}(x)$:

$$(1') \qquad\qquad\qquad \nabla \cdot \underline{A}_\ell = G_\ell \qquad \ell = 1, 2, \ldots$$

$$(2') \qquad \begin{cases} \nabla \times \underline{u} = \underline{W} & \quad (2a') \quad \nabla \cdot \underline{W} = 0 \\ \nabla \times \underline{v} = \underline{\Omega} & \quad (2a'') \quad \nabla \cdot \underline{\Omega} = 0 \end{cases}$$

where $\underline{A}_\ell = \underline{A}_\ell(\underline{x}, \underline{u}, \underline{v})$, e.g. the electromagnetic field equations.

The system of equations (1), (2) is quite general, it can be

linear and nonlinear, elliptic, hyperbolic or mixed, with smooth

or non smooth solutions. The independent variables x_i may desig-

nate space and time coordinates and different kinds of initial and/or

boundary conditions may be appropriate for different problems. Higher
order equations may be put into this form, applications include problems
from electro-magnetic field theory, fluidynamics, and plasma-dynamics,
including flows with shocks.

Initial and Boundary Conditions.

A unique solution to a field governed by equations (1), (2) is depicted for appropriate initial/boundary conditions , where a numer (possible all) of the field components u_i are given on parts (possibly all) of the boundary $\partial\Omega = \sum_{j=1}^{N} \partial\Omega_j$ of the region Ω occupied by the field. i.e.

$$(3) \qquad \tilde{u}_i(x) = \underline{f}_i(x) \quad \text{on} \quad \partial\Omega_\ell \qquad 1 \leq \ell \leq N.$$

The specific form of (3) depends on the problem and on the type of the partial differential system (1), (2) (e.g. pure boundary values for time independent elliptic problems, initial values for the Cauchy hyperbolic case).

3. Redundancy and Reduction.

The system (1), (2), includes $(n + 1)$ first order partial differential equations for the m unknown components of the dependent vector $\underline{u}$. The n equations (2), however, are not linearly independent. In order to solve a problem, the system has to be reduced to a 'posed' system, a necessary condition that is the number of equations equal to the number of unknowns.

(a) Potential formulation.

For smooth solutions, one has traditionally defined auxialliary functions: the scalar and vector potentials, which for the system (1), (2) can be written as:

$$(4) \qquad \underline{A} = - \nabla \times \underline{\lambda} + \underline{g}(\phi)$$

$$(5) \qquad \underline{u} = \nabla \phi + \underline{f}(\underline{X})$$

where $\underline{\lambda}(\underline{u},x)$ is a vector potential, ϕ is a scalar potential and: g and f are any solutions of:

$$(6) \qquad \nabla \circ \underline{g} = G, \quad \nabla \times \underline{f} = \underline{W}$$

Upon substituting (4) and (5) in (1), (2) one gets a 2nd order system of 4 equations for the 4 components of the scalar and vector potentials (ϕ, λ).

The redundancy in the first order system is exchanged for a redundancy of a different kind in the higher order system, the latter admitting a family of superfluous solutions, all describing the same field. This is embodied in the so-called gauge invariance for the common scalar and vector potential representation, and the

actual fields (unique for determinate well posed problems) are obtained with the aid of additional gauge conditions and an appropriate consideration of initial and boundary conditions if needed.

In classical applications, where the system is deterministic and all field quantities can be measured directly the field-potential correspondence is clear and unique. The potentials, although providing a very useful and elegant framework can nontheless be considered an auxiliary mathematical entities and the gauge transformations a mathematical artifact, while the true physical information is completely contained in the fields themselves. Potential representation has been immensely useful in theoretical physics and continuum mechanics; however, the primitive field variables are sometimes preferable especially for rotational fields and for flows with shocks, for numerical simulations based on both finite differences and finite elements descretizations.

(b)   Addition of a new variable.

A suggestion, to equalize the number of equations and unknowns and still solve directly for the fields is to add a new dependent variable:

$$v = \nabla \cdot \underline{u}$$

and using the relation:

$$\nabla \times \nabla \times \underline{u} = \nabla(\nabla \cdot \underline{u}) - \nabla^2\underline{u}$$

replace equation (2) by its rotor:

$$(7) \qquad \nabla^2\underline{u} = \nabla v - \nabla \times \underline{W},$$

which, with equation (1) gives $(n + 2)$ equations for the $(n + 2)$ unknowns $(\underline{u},v)$.

The formulation above has been suggested (and used) by M. Mock for computational purposes, with a staggered mesh for (u,v) (to avoid decoupling of the descretized equations for $\underline{u}$ and v) to solve boundary value problems. This formulation is not directly extendable to the unsteady case.

Direct field formulation.

The auxiliary function formulations invariably raise the order of the equations to be solved; the first-order system becomes second-order. This requires a higher degree of smoothness for the solution function and may be a draw-back for numerical analysis and calculations. Thus although many large computer simulations are based on 'potential' formulations, a direct solution of the first order system has been found beneficial [5], sometimes essential [9], especially for 'initial' rather than boundary value problems, and for non-potential flows (e.g. Euler solvers in aerodynamics).

The question is how to choose the 'right' n rotationality conditions that will give with the continuity equation (1), the 'right' $(n + 1) \times (n + 1)$ system. A simple treatment for irrotational, 3-dimensional fields (worked out for the problem in [5]), is given in [13]. It is extended here for the general case (equation (1), (2)), where the field may have sources and be rotational (e.g. an electromagnetic field with moving charges, flow behind a curved shock, motion of reacting gases.)

Choice of Equations for 3 space-Dimensions.

Consider the case of 3 dependent and 3 independent variables
(e.g. Cartesian space coordinates).[†]

We get:

$$\underline{x} = (x_1, x_2, x_3) = (x, y, z)$$

$$\underline{u} = (u_1, u_2, u_3) = (u, v, w)\,(x, y, z)$$

$$\underline{A} = (A^{(1)}, A^{(2)}, A^{(3)})\,(u, v, w;\ x, y, z)$$

and the system of equations is:

(1) $\qquad A_x^{(1)} + A_y^{(2)} + A_z^{(3)} = G$

and:

(2) $\qquad \begin{cases} \text{(i)} \qquad w_y - v_z = W^{(1)} \\[2ex] \text{(ii)} \qquad u_z - w_x = W^{(2)} \\[2ex] \text{(iii)} \qquad v_x - u_y = W^{(3)} \end{cases}$

The system (1), (2) contains 4 equations for the three unknown functions (u, v, w), a redundant system for a well defined field.

In addition:

(2a) $\qquad\qquad\qquad W_x^{(1)} + W_y^{(2)} + W_z^{(3)} = 0.$

[†] This treatment can be directly extended to time-dependent and larger systems. For example, for the three dimensional, time dependent case, equation (1) has an additional term, with an additional equation (e.g. an energy equation).[‡] Initial conditions are given everywhere. Equations (2) remain the same and the whole treatment carries over.

However, for twice continuously differentiable (u,v,w) and once continuously differentiable $(W^{(1)}, W^{(2)}, W^{(3)})$ we get by differentiating, equating mixed derivatives, and substituting:

$$\frac{\partial}{\partial z}(iii) : v_{xz} - u_{yz} = W_z^{(3)}$$

substituting u_z from (ii):

$$vx_z - w_{xy} - W_y^{(2)} = W_z^{(3)}$$

$$(v_z - w_y)_x = W_y^{(2)} + W_z^{(3)}$$

using (2a):

$$(w_y - v_z)_x = W_x^{(1)}.$$

and integrating with respect to x:

$$w_y - v_z = W^{(1)}(x,y,z) - W^{(1)}(x_0,y,z)$$

Thus equation (i) is implied by equations (iii) and (ii) provided that $W^{(1)}$ is given at $x = x_0$ for all (x,y).

In the same manner each two of the equations determine the third, for which appropriate initial conditions have to be given.

Algebraic treatment.

Written in a matrix form, equations (1), (2) are:

$$(8) \quad \begin{bmatrix} A_u^{(1)} & A_v^{(1)} & A_w^{(1)} \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}_x + \begin{bmatrix} A_u^{(2)} & A_v^{(2)} & A_w^{(2)} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}_y + \begin{bmatrix} A_u^{(3)} & A_v^{(3)} & A_w^{(3)} \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}_z$$

$$= \begin{bmatrix} G - \sum_{i=1}^{3} \dfrac{\partial A^{(i)}}{\partial x_i} \\ w^{(1)} \\ w^{(2)} \\ w^{(3)} \end{bmatrix}$$

or:

$$(8a) \qquad C^{(x)} \underline{u}_x + C^{(y)} \underline{u}_y + C^{(z)} \underline{u}_z = \underline{F}$$

with the corresponding (4×3) matrices C and 4 forcing functions F.

The system (8) is again redundant, and one of the last 3 equations has to be deleted to render (4×3) coefficient matrices C into (3×3) matrices $\tilde{C}$. The choice is directed by the marching "time like" direction for which initial conditions are given, for which the $\tilde{C}^{(i)}$ matrix has to be invertible, hence nonsingular, hence with no row (or column) of zeros. This automatically rules out one equation: when (u,v,w) are given at

$x = x_0$ (e.g. $x_0 \rightarrow -\infty$ for steady flow about an obstacle), the first irrotationality condition has to be omitted. A marching procedure along the x axis requires the inversion of the non-redundant matrix $\widetilde{C}(x)$ such that

$$(8b) \qquad \underline{u}_x = -[\widetilde{C}^{(x)}]^{-1}\left(\widetilde{C}^{(y)}\underline{u}_y + \widetilde{C}^{(z)}u_z\right) + [\widetilde{C}^{(x)}]^{-1}\underline{r}.$$

In the same manner, integration schemes along the y and z directions require the omission of (8i) and 8iii), respectively, as a necessary condition (not always sufficient!) for a well defined, stable scheme, (as is seen in [5]).

4. **Examples**.

(a) Rotational, steady aerodynamics.

Substituting for $\rho(u^2)$ in (1a) and eliminating the time derivatives one gets (e.g. [5]):

(10) $(a^2 - u^2)u_x + (a^2 - v^2)v_y + (a^2 - w^2)w_z - uw(w_x + u_z) - uv(u_y + v_x) - vw(w_y + v_z) = 0$

$$(i) \qquad w_x - u_z = w^{(1)} \qquad\qquad (6)$$

$$(ii) \qquad w_y - v_z = w^{(2)}$$

$$(iii) \qquad u_y - v_x = w^{(3)}$$

i.e. 4 equations for the 3 unknown functions (u, v, w).

The potential formulation has not been found feasible for rotational flows (except for the two dimensional case, where the vector potential is the stream function), and the first order system is solved directly for this case [5].

The matrix formutation for: $\underline{u} = (u, v, w)$,   includes:

$$C^{(x)} = \begin{pmatrix} a^2 - u^2 & -uv & -uw \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

and similarly $C^{(y)}$ and $C^{(z)}$,

$$F = (0, w^{(1)}, w^{(2)}, w^{(3)}).$$

and the equation containing the row of 0's in the marching direction (for which "initial" conditions are known) is to be eliminated.

(a1)   Irrotational, unsteady aerodynamics

We set:

$$\underline{x} = (x_0, x_1, x_2, x_3) = (t, x, y, z)$$

$$\underline{u} = (u_1, u_2, u_3) = (u, v, w)$$

$$\underline{A} = (\rho, \underline{u})$$

$$\rho = {}_0(u^2)$$

$$G = 0, \quad \underline{W} = 0$$

and get:

$$(1a) \qquad \rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0$$

$$(2a) \qquad \nabla \times \underline{u} = 0$$

$$\rho = \rho(u^2).$$

Deriving $\underline{u}$ from a potential $\phi : \underline{u} = \nabla \phi$ satisfied $(2a)$ identically. In addition, ϕ has to satisfy a 2nd order nonlinear equation derived from $(1a)$.

The first order system, in the matrix form $(3a)$ is:

$$u = (v, u, w), \quad F = (0,0,0,0)$$

$$C^{(x)} = \begin{bmatrix} \rho_u u + \rho & \rho_v u & \rho_w u \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

and similarly $C^{(y)}$ and $C^{(z)}$.

(b) Maxwells equations in vacuum.

The electromagnetic field equations are [1]:

(11) $\nabla \cdot \underline{H} = 0$ (13) $\nabla \times \underline{E} = -\frac{1}{c} \underline{H}_t$

(12) $\nabla \cdot \underline{E} = 4\pi\rho$ (14) $\nabla \times \underline{H} = \frac{1}{c} \underline{E}_t + \frac{4\pi}{c} \underline{j}$ $(\varepsilon = \mu = 1)$

Where $\underline{E}$, $\underline{H}$; ρ, $\underline{j}$ designate the electric and magnetic fields, the electric charge and electric current distributions, respectively. They are of the form (1a), (2a) where:

$$A_1 = \underline{H}, \quad \underline{A}_2 = \underline{E},$$

$$G_1 = 0, \quad G_2 = 4\pi\rho$$

$$W_1 = \frac{1}{c} \underline{E}_t + \frac{4\pi}{c} \underline{j}, \quad W_2 = -\frac{1}{c} \underline{H}_t$$

Equations (11) - (14) are not independent: for smooth fields (twice differentiable) equations (13) implies $(\nabla \cdot \underline{H})_t = 0$ and equation (11) holds for all times if it holds for $t = t_0$, thus it is but an initial condition (at most), that has to be satisfied by $\underline{H}$ at $t = 0$.

The electric charge density and currents (ρ and $\underline{j}$ respectively) cannot be prescribed arbitrarily, since equation (14), and (12) imply a constraint on their source terms ρ and $\underline{j}$:

(15) $\rho_t + \frac{1}{c} \nabla \cdot \underline{j} = 0$ (continuity of charge)

which has to be satisfied at all times.

From the accounting (and counting) point of view for 3 space dimensions, the system includes 8 equations for   unknowns

(three components for each $\underline{E}$ and $\underline{H}$). Equations (13) and (14) can be viewed as evolution equations, while (11) and (12) are mere constraints, augmented by the compatibility condition (15). This classification is certainly not unique and not always obvious (e.g. in the stationary case, where the time derivations are absent and the two pairs look equivalent).

Reduction to a minimal set.

(i) Potential formulation

For Maxwells equations the standard analysis is done via the scalar and vector potentials: ϕ and $\underline{A}$ respectively, (e.g. [11]), where:

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \nabla\phi \ , \quad \underline{H} = \nabla \times \underline{A}.$$

Written in terms of $(\phi, \underline{A})$ Maxwell's system reduce to 4 scalar equations for the 4 components of ϕ and $\underline{A}$.

The resulting equations are higher order, and admit a wider family of solutions, all equivalent under gauge transformations:

$$\phi' = \phi - \frac{1}{c}f_t(\underline{x}, t)$$

$$\underline{A}' = \underline{A} + \nabla f$$

or in 4 dimensional notation:

$$(\underline{A}, -\phi)' = (\underline{A}, -\phi) + [\nabla f, \ -\frac{1}{c}f_t].$$

The potential formulation for the electromagnetic field is endowed with a beautiful structure, lucid transformation properties and striking accessible information content (e.g. decoupling into 2nd order wave equations for each of the components) unfolding the wealth of electromagnetic waves reference [1].

(ii)   Adding new variables

Setting:

(16) $h = \nabla \cdot \underline{H}$, $e = \nabla \circ \underline{E}$

to be added to the system (11) - (15) gives the additional depend-
ent variables required, and, again, a second order system obtained
by substituting the expressions (15) in (13) and (14). The
system (11) - (16) the same number of equations and unknowns, at the
cost of 2 additional dependent variables and a 2nd order equations.

(iii) Matrix formulation

Equations (11) - (14) when spelled out (in Cartesian coord-
inates, say) can be readily put in a matrix form (4),(5). The
criterion of a non-singular coefficient matrix in the marching
direction applies, resulting in an appropriate elimination of the
redundant equations. No new variables are added, and the system
remains first order, hence admitting non-smooth solutions and the
treatment of nonlinear problems. Combined electromagnetic-fluid-
dynamic flows, as well as other field combinations can be formulated,
reduced and discretized accordingly.

II. ALTERNATIVE VARIATIONAL FORMULATIONS.

A variational formulation of the field $\underline{u}(x)$ satisfying (1), (2), (3) is a functional $J(\underline{v})$ defined on Ω whose stationary value is obtained for $\underline{v} = \underline{u}$:

$$\delta J(\underline{v}) = 0 \Leftarrow \underline{v} = \underline{u} \; .$$

For well posed problems, for which $\underline{u}(\underline{x})$ is unique

$$\delta J(\underline{v}) = 0 \Longleftrightarrow \underline{v} = \underline{u}.$$

Variational formulations are scalar, short, additive (the functionals for complex systems are direct sums of their simpler parts), invariant under appropriate classes of transformations and are often convenient for theoretical analysis and for numerical simulations, e.g. by finite differences and finite elements [10], [11], [12]; the integrals can easily be descritized and approximated, and smoothness requirements on the functions are less stringent than for the corresponding differential system. This last point is most important from the numerical view point and for the treatment of shocks.

The case $G = 0$ is described in [16]. The functional $J(\underline{v},\underline{x})$:

$$(20) \qquad J = \int_{\Omega} [\; U(\underline{v},\underline{x}) + \underline{\lambda} \, (\nabla x \underline{v} - \underline{w})] dx + \int_{\partial \Omega_m} \underline{\lambda} x \underline{v} \cdot d\underline{\sigma}$$

is stationary at $\underline{v} = \underline{u}$:

$$(21) \qquad \qquad \delta J(\underline{v}) = 0, \; \underline{v} = \underline{u}$$

provided that:

(22)
$$\nabla_{\underline{u}} \times \underline{A} = 0.$$

(23)
$$\underline{A} = \nabla_{\underline{u}} L$$

resulting in:

(24)
$$\underline{A} = -\nabla \times \underline{\lambda}.$$

The variation is done on all $\underline{v}$ for which J is defined, coercive boundary conditions are satisfied on the boundary $\partial\Omega_i = \partial\Omega - \partial\Omega_m$, or for which $\underline{\lambda}\|\underline{v}$ or $\underline{v}\|\underline{d\sigma}$ on $\partial\Omega$,

For the non-sourceless (or sourceful) case, the following variational statements hold:

Theorem 1. The functional:

(25)
$$J(\underline{v},\underline{\lambda}) = \int_\Omega [L - \underline{g} \cdot \underline{v} + \underline{\lambda} \cdot (\nabla \times \underline{v} - \underline{W})] \cdot d\underline{x} + \int_{\partial\Omega_m} \underline{\lambda} \times \underline{v} \cdot \underline{d\sigma}$$

is stationary for $\underline{v} = \underline{u}$ satisfying (1), (2), (3) provided that

(26)
$$\nabla_{\underline{u}} \times \underline{A} = 0$$

where:

(27)
$$\nabla \cdot \underline{g} = G \quad \text{or} \quad \underline{g} = \nabla^{-1}G$$

(28)
$$\underline{A} = \nabla_{\underline{u}} L$$

(29)
$$\underline{A} = \nabla \times \underline{\lambda} + \underline{g}$$

$\underline{v}$ is allowed to vary over all functions for which (25) is defined and finite, and which satisfy the coercive B.C. (3). In this formulation $\underline{v}$ is required to be differentiable while $\underline{\lambda}$ can be just integrable.

Necessary conditions for stationarity of $J(\underline{v},\underline{\lambda})$, i.e.

(21a) $$\delta J = 0 \quad \text{at} \quad \underline{v} = \underline{u}$$

are:

$$\nabla_{\underline{u}} L - \underline{g} = - \nabla \times \underline{\lambda}$$

i.e.,

(28) $$\underline{A} = \nabla_{\underline{u}} L, \quad \nabla \cdot \underline{A} = \nabla \cdot \underline{g}$$

(29) $$\underline{A} = - \nabla \times \lambda + \underline{g}$$

and in addition:

$$\int \delta \underline{\lambda} \times \underline{v} \cdot d\sigma = 0$$

on the free boundary.

Remark. Natural boundary conditions for the functional (25) without the surface term are: $\underline{\lambda} = 0$ or $\underline{\lambda} \parallel d\sigma$ on the 'free' boundary (where $\underline{v}$ is not prescribed).

Proof.

An arbitrary variation of $\underline{v}$ and $\underline{\lambda}$ in $J(\underline{v},\underline{\lambda})$ (25) leads to:

$$\delta J = \int_{\Omega} [\delta \underline{L} - \underline{g} \cdot \delta \underline{v} + \delta \underline{\lambda} \cdot (\nabla \times \underline{v} - W) + \underline{\lambda} \cdot \nabla \times \delta \underline{v})] d\underline{x}$$
$$+ \int_{\partial \Omega} \delta \underline{\lambda} \times \underline{v} \cdot d\sigma + \int_{\partial \Omega} \underline{\lambda} \times \delta \underline{v} \cdot \underline{d\sigma}.$$

Substituting:

$$\delta L = \nabla_{\underline{v}} L \cdot \delta \underline{v}$$

$$\underline{\lambda} \cdot \nabla \times \underline{v} = \delta \underline{v} \cdot \nabla \times \underline{\lambda} - \nabla \cdot (\underline{\lambda} \times \delta \underline{v})$$

we get:

(25a) $$\delta J = \int_{\Omega} [(\nabla_{\underline{v}} L - \underline{g} + \nabla \times \underline{\lambda}) \cdot \delta \underline{v} + (\nabla \times \underline{v} - W) \cdot \delta \underline{\lambda}] d\underline{x}$$
$$- \int_{\partial \Omega} \underline{\lambda} \times \delta \underline{v} \cdot d\underline{\sigma} + \int_{\partial \Omega} \delta \underline{\lambda} \times \underline{v} \cdot d\sigma + \int_{\partial \Omega} \underline{\lambda} \times \delta \underline{v} \cdot d\underline{\sigma}.$$

An alternative formulation is obtained by integrating the second term by parts, using the vector identity:

$$\nabla \cdot (\underline{\lambda} \times \underline{v}) = \underline{v} \cdot \nabla \times \underline{\lambda} - \underline{\lambda} \cdot \nabla \times \underline{v}$$

substituting in (25) $\underline{\lambda} \cdot \nabla \times \underline{v}$, we obtain:

Theorem 2. The functional

$$(30) \qquad J(\underline{v}, \underline{\lambda}) = \int (L - \underline{g} \cdot \underline{v} + \underline{v} \cdot \nabla \times \underline{\lambda} - \underline{\lambda} \cdot \underline{W}) \underline{dx}$$

is stationary for $\underline{v} = \underline{u}$ satisfying (1), (2).

In the variational formulation (30), the Lagrange multiplier $\underline{\lambda}$ is required to have integrable first derivatives, which appear explicitly in J, while $\underline{v}$ can just be integrable.

The surface term drops out, and the solution $\underline{v} = \underline{u}$ satisfies the natural boundary conditions:

$$\underline{\lambda} \times \underline{u} \cdot \underline{d\sigma} = 0 \quad \text{on} \quad \partial\Omega, \quad \text{i.e.,}$$

either

$$\underline{\lambda} \parallel \underline{u} \quad \text{or} \quad \underline{u} \parallel \underline{d\sigma} \text{ i.e., } \underline{u} \parallel \hat{n} \quad \text{on} \quad \partial\Omega.$$

Examples:

a)   Steady electromagnetic field.

Maxwells' equations for a steady state can be written as [3]:

i) $\nabla \cdot \underline{E} = \rho$

ii) $\nabla \times \underline{E} = 0$

iii) $\nabla \cdot \underline{B} = 0$

iv) $\nabla \times \underline{B} = \underline{j}$

A variational statement for i), ii) is:

$$(25') \quad J(\underline{E}, \underline{\lambda}^E) = \int_\Omega [(E^2/2 - \underline{g} \cdot \underline{E}) + \underline{\lambda}^{(E)}(\nabla \times \underline{E})]dx + \int_{\partial\Omega_i} \underline{\lambda}^{(E)} \times \underline{E} \cdot \underline{d\sigma}$$

for $\underline{E}$ arbitrary, or:

$$(30') \quad J(\underline{E}) = \int_\Omega (E^2/2 - \underline{g} \cdot \underline{E})dx \text{ for E irrotational}$$

where: $\underline{g}$ is any solution to

$$\nabla \cdot \underline{g} = \rho \quad \text{or} \quad \underline{g} = \nabla^{-1}\rho.$$

The corresponding variational formulations for the magnetic equations iii), iv) are:

$$(25'') \quad J(\underline{B}, \underline{\lambda}^B) = \int [B^2/2 + \underline{\lambda}^{(B)}(\nabla \times \underline{B} - \underline{j})]dx + \int_{\partial\Omega_m} \underline{\lambda}^{(B)} \times \underline{B} \cdot \underline{d\sigma}$$

and

$$(31') \qquad J(\underline{B}) = \int_{\Omega} [B^2/2 - \underline{B} \cdot (\nabla \times \underline{\lambda}^{(B)})]$$

the functionals for the combined field are obtained by simply adding the ones for the 'separated' system:

$$(25''') \quad J(\underline{E}, \underline{B}; \underline{\lambda}^E, \underline{\lambda}^B) = \int_{\Omega} \left[\frac{E^2 - B^2}{2} - \underline{g} \cdot \underline{E} + \lambda^B \cdot \underline{j} + \underline{\lambda}^{(E)} \cdot (\nabla \times \underline{E}) - \underline{\lambda}^{(B)} \cdot (\nabla \times \underline{B}) \right] dx$$

$$+ \int_{\partial \Omega_m^E} \underline{\lambda}^E \times \underline{E} \cdot \underline{d\sigma} - \int_{\partial \Omega_m^B} \underline{\lambda}^B \times \underline{B} \cdot \underline{d\sigma} .$$

Finally $(30')$ and $(31')$ combine to give:

$$(31'') \quad J(\underline{E}, \underline{B}; \underline{\lambda}^E, \underline{\lambda}^B) = \int \left[\frac{E^2 - B^2}{2} - \underline{g} \cdot \underline{E} - \underline{E} \cdot \nabla \times \underline{\lambda}^E + \underline{B} \cdot \nabla \times \underline{\lambda}^B \right] .$$

b.   **Steady fluidynamics.**

The differential equations are:

$$\nabla \cdot (\rho \underline{q}) = 0$$

$$\rho = \rho(q^2)$$

$$\nabla \times \underline{q} = \underline{W}$$

and the corresponding Lagrangian L and functionals are:

(25)
$$\rho \underline{u} = \frac{\partial L}{\partial \underline{u}} \quad \text{or:} \quad \rho u_i = \frac{\partial L}{\partial u_i}$$

$$J(\underline{u}) = \int_{\Omega} [L + \underline{\lambda} \cdot (\nabla \times \underline{u} - \underline{w})] + \int_{\partial\Omega} \underline{\lambda} \times \underline{u} \cdot d\sigma$$

or

(31)
$$J(\underline{u}) = \int_{\Omega} L - \underline{\lambda} \cdot \underline{w} + \underline{u} \cdot \nabla \times \underline{\lambda}$$

for the appropriate function spaces, with the required smoothness

properties and constraints, in the region and/or on the boundary.

The results (25) have been described and analyzed in [16],[17]

where applications to specific flows are given. While (25)

requires that $\underline{u}$ be smoother than $\underline{\lambda}$, the opposite holds for (31),

which is more in accord with the physics of the problem, where the

field variables $\underline{u}$ are related to the derivatives of the 'vector pot-

ential' $\underline{\lambda}$.

Redundancy in Variational Formulations.

Variational formulation have been widely used for approximating field solutions, notably by the Ritz and finite elements methods. The tacit assumption is that the variational problem is well posed, for which a necessary condition is that so is its system of Euler equations and initial boundary conditions; a necessary condition in turn, for stable solvability is that it be non-redundant, as shown in part I above.

The variational formulations Eq. (25), (30) are redundant in the sense that their Euler Equations (1), (2) are redundant. Since one of the rotationality conditions (2) is implied by the rest, its corresponding λ component is completely redundant. This can also be inferred from Eq. (29) which is analogous to (2) (with $\underline{u}$ replaced by $\underline{\lambda}$ and $\underline{w}$ by $(\underline{A} - \underline{g})$), where the system (29) is not independent, and one of the equations results from the other ones.

This redundancy in the variational formulations is less apparent than in the differential system (where a simple count of equations and unknowns signals a red light - to stop and look). In the usual approximations (e.g., by the Ritz or finite element method) the number of equations is made to equal the number of unknowns, obscuring the fact that the system may have no solution (for incompatible $\underline{w}$ not satisfying Eq. (2a)). Simple round-off inaccuracies may turn various procedures unstable. This, we believe may be one of the causes for instability experienced in various mixed and hybrid finite elements procedures (e.g. [10] - [14]).

Two remedies are suggested:

(i)   to remove one rotationality condition, the one in the time-like direction (or mostly so for nonlinear problems)

(ii)   to add another term to the functional:

$$\delta \nabla \cdot \underline{w}$$

to enforce the compatibility of the $\underline{w}$ components and correct any deviation thereof, while (i) removes one of the unknowns (but demands attention in choosing the correct one to eliminate) (ii) adds an unknown - another Lagrange multiplier which is a draw back. The scheme, however, may be used as a check on simple bench-mark problems.

Concluding Remarks.

The variational formulations in primitive variables hold for a wide class of multi-dimensional nonlinear systems and are convenient to descretize for arbitrarily complex geometries. They also hold promise for treating multi-dimensional shocks as actual jumps with correct jump conditions (which most other methods cannot do), by treating the integrand in the functional as piece-wise continuous with an appropriate generalization of the Erdmann conditions at the jump front. This as well as further stability consideration for different variational problems and the relation to other variational formulations in primitive variables (e.g. [19]) are to be treated in a following report.

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APPENDIX

The simplest redundant system.

The redundancy in the rotationality equations (2) seems somewhat analogous to the trivially simple case of 3 equations with 2 unknowns:

$$\text{L1)} \quad a_{11}x + a_{12}y = b_1$$

$$\text{(L)} \quad \text{L2)} \quad a_{21}x + a_{22}y = b_2$$

$$\text{L3)} \quad a_{31}x + b_{32}y = b_3$$

The first 2 equations with 2 unknowns (a 2×2 system)

$$\text{L1)} \quad a_{11}x + a_{12}y = b_1$$

$$\text{L2)} \quad a_{21}x + a_{22}y = b_2$$

may have:

a) a unique solution (x,y) if and only if the equations are linearly independent, i.e.,

$$\text{(Aa)} \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0.$$

b) a one parameter family of solutions $(x(s), y(s))$ if they are linearly dependent, i.e.

$$\text{(Ab)} \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = 0$$

or:

$$(a_{21}, a_{22}, b_1) = \alpha(a_{11}, a_{12}, b_1)$$

which means that one equation is a simple multiple of the other, and

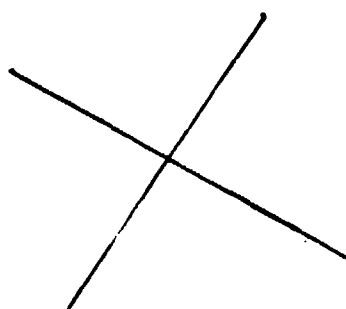$$y = -\frac{a_{11}}{a_{12}} x + \frac{b_1}{a_{12}} = -\frac{a_{21}}{a_{22}} x + \frac{b_2}{a_{22}}$$

which may be parametrized:

$$(x,y) = (s, as + b); a = -\frac{a_{11}}{a_{12}} = -\frac{a_{21}}{a_{22}}; \quad b = \frac{b_1}{a_{12}} = \frac{b_2}{a_{22}}.$$
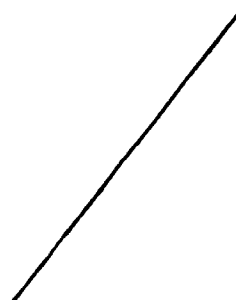
c) no solution if

(Ac)
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0, \quad \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \neq 0.$$
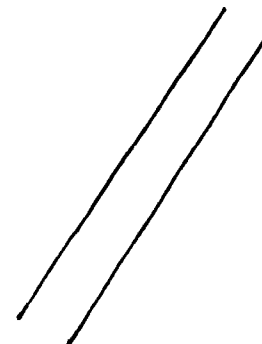
Geometrically the equations models lines in the plane, and the 3 cases are:

(a)   (b)   (c)

The case (a) is the usual (or more prevalent one) while (c) and (b) are exceptional.

With the added equation L_3 the (3×3) system (L) the system may have

A) a unique solution if

$$\frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{\begin{vmatrix} b_2 & a_{22} \\ b_3 & a_{32} \end{vmatrix}}{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}$$

and

$$\frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{\begin{vmatrix} a_{21} & b_2 \\ a_{31} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}$$
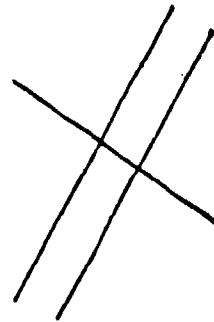
B) a one parameter family of solutions if all three equations are linearly interdependent, or any 2×2 sub-determinants for the 4×3 matrix $\{a_{ij}, b_i\}$ is zero, i.e., at least 2 of the equations are multiples of each other.

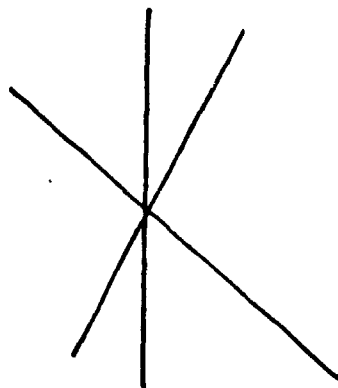C) no solution, when none of the conditions above holds.

Geometrically the possibilities are:
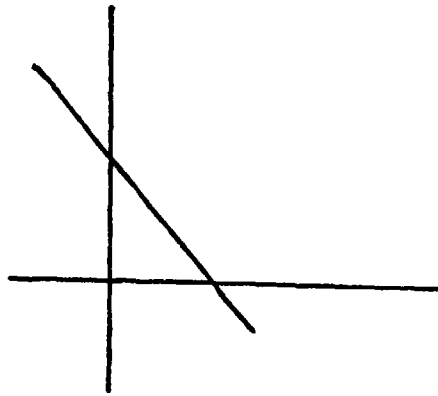
C

C'

A

B

A'

i.e.

A: the three lines L_1, L_2, L_3 intersect at one point.

A': two lines (L_1, L_2) coalesce, intersecting L_3 at a point.

B: three coalscing lines $L_1, L_2, L_3 \to L : (x,y)(s)$.

C: three lines intersecting at 3 points - the most prevalent
    situation of no solution to the system.

The case discussed in the paper is analogous to the case A, where
presumably there exists a unique solution, satisfying the too many, but
congruent equations "meeting at one point". The problem is that due
to inaccuracies, the equations are not exact, which immediately shifts
us to the situation C of no solution.

This simple model of redundancy readily extends to larger "rectangu-
lar' (rather than 'square') systems, with their geometrical representa-
tion.

The 4 linear equations with 3 unknowns:

(P) $\quad a_{ij}x_j = b_j$, $\quad i = 1,\ldots,4$, $\quad j = 1,2,3$.

describe 4 planes $(P_1,\ldots,P_4)$ in 3 dimensional space.

Depending on the (4×3) matrix of coefficients a_{ij} and the vector b_i, or, equivalently, the matrix $\{a_{ij},b_i\}$, the planes may:

I. Coalesce to one plane (if all the equations are linearly dependent).

II. Intersect along 1 to 4 lines

III. Meet at one point, which forms a unique solution $\{x_j\}$, $j = 1,2,3$.

This last case is the most special and requires special compatibility conditions, which are violated by round-off errors even for supposedly compatible systems.

SADDLE POINT APPROXIMATION FOR THE

TRICOMI EQUATION

by

Sara Yaniv, Frieda Loinger and Nima Geffen

Abstract

The Tricomi equation is solved in a mixed elliptic-hyperbolic domain
by a finite-element implementation of a new variational principle. The
nonuniformly elliptic part is solved by using a projection of the boundary
condition in the hyperbolic region onto the parabolic line. This leads to
a convergent solution which is extended into the hyperbolic part with
characteristic finite elements by solving either a Goursat or a Cauchy
boundary value problem.

1. INTRODUCTION

The Tricomi equation, one of the simplest equations of mixed ellip-
tic-hyperbolic type and of interest in physical applications, representing
small perturbations transonic flows in the hodograph plane, has been inves-
tigated extensively, e.g. [1]. The Tricomi problem, where boundary data
are given on an elliptic arc and one characteristic in the hyperbolic half
plane, can be separated into a nonuniformly elliptic problem, where bound-
ary data are given on the elliptic arc and the parabolic segment 'closing'
it, and a nonuniformly hyperbolic triangle bounded by a parabolic "basis"
and two characteristics emanating from its sides (figure 1). Other bound-
ary conditions have also been treated [1].

An analysis of a complete linear boundary value problem of whatever
type and boundary conditions is given in [2], with conditions for well
posedness and a weak formulation. The last analysis is constructive and
directly amenable to discretization of a mixed elliptic-hyperbolic domain,
using the same procedure in both parts, in a Galerkin-type weak formulation.
A finite-difference scheme based on this weak formulation is used in [6]
with convergence $O(h^{1/2})$ and triangular linear finite elements, with
convergence $O(h^2)$ in [7], where a comprehensive survey of methods and
an extensive bibliography are also given. The convergence reported in
[7] is better than the theoretical estimates in [3] for the same problem.

A direct finite-element analysis and estimates of the nonuniformly ellip-
tic part is given in [4]. Finite-elements calculations using patched
variational principles are given in [12] with "experimental" second order
accuracy for the tested problems. Finite difference calculations have
also been tried, e.g. [5].

Another mixed type equation solved recently is the Lavrent'ev-
Bitsadze equation [14], where the boundary condition on the hyperbolic
region are transferred to the parabolic segment and the equation in the
elliptic region is solved by the finite element method.

In (15) a weak formulation for the problem based on different
spaces of test and trial functions is used to prove existence,
uniqueness and uniform stability of the approximate solution.

2. A VARIATIONAL FORMULATION

Tricomi's equation:

$$y\phi_{xx} - \phi_{yy} = 0 \qquad (1)$$

is elliptic in the lower half plane $y < 0$ and hyperbolic in the upper
one, $y > 0$ (figure 1), where it's real characteristics are:

$$\xi = x + \frac{2}{3} y^{3/2} \qquad (2a)$$

$$\eta = x - \frac{2}{3} y^{3/2} . \qquad (2b)$$

Writing equation (1) as a first order system, one gets:

$$(u,v) = (\phi_y, \phi_y)$$

$$(yu)_x - v_y = 0 \qquad (1a)$$

$$u_y - v_x = 0 . \qquad (1b)$$

We now look for a solution in a mixed domain $\Omega = \Omega_1 \cup \Omega_2$ where
Ω_1 is in the lower ($y<0$) elliptic half plane with its boundary $\partial\Omega_1$
and Ω_2 in the upper ($y>0$) hyperbolic one with it's boundary $\Gamma_1 \cup \Gamma_2$
where Γ_1 and Γ_2 are two characteristics with boundary conditions given
along $\partial\Omega_1$ and one of the characteristics, say Γ_1 (figure 1).[+]

[+] Other domain shapes with different conditions can be similarly considered.

2.1 A variational formulation in the elliptic region.

First, let us consider the Dirichlet problem:

$$y\phi_{xx} - \phi_{yy} = f(x,y) \tag{3a}$$

in a domain Ω ($y<0$) with given boundary conditions:

$$\phi(x,y)\Big|_{\partial\Omega} = 0 \tag{3b}$$

Assuming

$$u = \phi_x$$
$$v = \phi_y$$

then the operator $(uy)_x - v_y$ is a potential [13, page 35].

The variational formulation for the problem is: find $(u,v;\lambda)$ so that

$$J(u,v;\lambda) = \int_{\Omega}\int [L(x,y,u,v) + \lambda(u_y - v_x)]dxdy \tag{4}$$

$$- 2\int_{\Omega}\int u\cdot F(x,y)dxdy$$

where

$$F(x,y) = \int^x f(\xi,y)d\xi.$$

is stationary, for all functions $(u,v) \in U\times V$ and $\lambda \in W$.

$L(x,y,u,v)$ is the Lagrangian of the problem, i.e.

$$L_u = yu$$

$$L_v = -v$$

and $\lambda(x,y)$ is a Lagrange multiplier which is the stream function of the problem [13, page 39].

The variation of $J(u,v;\lambda)$ for fixed λ gives the following weak formulation:

$$\delta J(u,v;\lambda) = \int_\Omega\int [yu\delta u - v\delta v + \lambda(\delta u_y - \delta v_x)]dxdy \tag{5}$$

$$- \int_\Omega\int \delta u \cdot F(x,y)dxdy = 0$$

Adding the equation

$$u_y - v_x = 0$$

in weak formulation:

$$\int_\Omega\int q(u_y - v_x)dxdy = 0 \quad for \quad \forall q \in W,$$

we get a saddle-point problem:

$$\int_\Omega\int [yuu_1 - vv_1 + \lambda(\frac{\lambda u_1}{\partial y} - \frac{\partial v_1}{\partial x})]dxdy = \int_\Omega\int u_1 F \; dxdy \tag{6a}$$

$$\forall(u_1,v_1) \in U \times V$$

and

$$\int_\Omega \int q \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) dxdy = 0 \quad \forall q \in W \tag{6b}$$

In the lower half plane $yu^2 - v^2 \leq 0$, hence

$$U \times V = \{(u,v) / \int_\Omega \int [-yu^2 + v^2 + (u_y - v_x)^2] dxdy < \infty,$$

$$udx + vdy \bigg|_{\partial\Omega} = 0 \},$$

$$\|(u,v)\|_{U \times V}^2 = \int_\Omega \int [-yu^2 + v^2 + (u_y - v_x)^2] dxdy$$

and, since λ depends on an arbitrary constant:

$$W = \{q | \int_\Omega \int q^2 dxdy < \infty , \int_\Omega \int qdxdy = 0 \}$$

$$|q|^2 = \int_\Omega \int q^2 dxdy$$

This formulation gives a saddle-point problem [9], which is determined as follows:

Given $f \in V'$, $g \in W'$, find $u \in V$, $\psi \in W$ such that:

$$a(u,v) + b(v,\psi) = f(v) \qquad \forall v \in V$$

$$(\text{7})$$

$$b(u,\phi) = g(\phi) \qquad \forall \phi \in W$$

Brezzi's theorem states the following conditions for existence and uniqueness of the solution:

Suppose a and b are bounded and

$$\inf_{\substack{u \in z \\ \|u\|_V = 1}} \sup_{\substack{v \in z \\ \|v\|_V = 1}} |a(u,v)| \geq \gamma > 0 \qquad (8a)$$

$$z \equiv \{u \in V / b(u,\phi) = 0 \quad \forall \phi \in W\}$$

$$\sup_{v \in V} \frac{|b(v,\psi)|}{\|v\|_V} \geq k \, \|\psi\|_W \qquad \forall \psi \in W, \quad k > 0 \qquad (8b)$$

then the solution of (5) is unique and

$$\|u\|_V + \|\psi\|_W \leq C(\|f\|_{V'} + \|g\|_{W'}).$$

It is easy to verify that problem (6a), (6b) satisfies (8a), (8b), hence has a unique solution which is the only solution of (3a), (3b).

In order to approximate the solution of problem (6a), (6b) using the saddle-point weak formulation or the saddle-point variational principle, one should use finite-dimensional spaces $V_h \subset V$ and $W_h \subset W$ satisfying:

$$\inf_{\substack{u \in Z_h \\ \|u\|_V = 1}} \sup_{\substack{v \in Z_h \\ \|v\|_V = 1}} |a(u,v)| \geq \tau > 0 \ , \ \tau \text{ independent of } h \qquad (9a)$$

$$\sup_{v \in Z_h} |a(u,v)| > 0 \quad \forall \ 0 \neq u \in Z_h \qquad (9b)$$

$$Z_h \equiv \{u \in V_h; \ b(u,\phi) = 0 \ \forall \phi \in W_h\}$$

$$\sup_{v \in V_h} \frac{|b(v,\psi)|}{\|v\|_V} \geq \ell \ \|\psi\|_W \qquad (9c)$$

$$\forall \psi \in W_h$$

$$\ell > 0, \quad \text{independent of } h.$$

The approximated problem is:

$$a(u_h,v) + b(v,\psi_h) = f(v) \qquad \forall v \in V_h$$

$$b(u_h,\phi) = g(\phi) \qquad \forall \phi \in W_h$$

$$(10)$$

Under hypothesis (9a), (9b), (9c) problem (10) has a unique

solution [9] and:

$$\|u - u_h\|_V + \|\psi - \psi_h\|_W \le c \inf_{\substack{\chi \in V_h \\ \delta \in W_h}} (\|u - \chi\|_V + \|\psi - \delta\|_W)$$

Our aim is to solve the Tricomi equation in a mixed domain,

hence we consider the integral relation along the parabolic line:

$$u(x,0) = \frac{d}{dx} \{\frac{1}{2\pi\gamma_1} x^{5/6} \frac{d}{dx} \int_0^x \frac{\psi(t/2)}{t^{2/3}(x-t)^{1/6}} dt$$

$$-\gamma \int_0^x \frac{v(t,0)}{(x-t)^{1/3}} dt\}$$

(11)

where

$$\psi(x) = \phi(y(x),x) \quad \text{along} \quad \Gamma_1, \quad \gamma_1 \quad \text{and} \quad \gamma$$

are given constants.

Condition (11) is a projection of the boundary conditions along

Γ_1 onto the parabolic segment via the Riemann invariants of the prob-

lem [1].

The problem to be solved is the Tricomi equation for $y \le 0$ with

Dirichlet conditions along $\partial\Omega_1$ and condition (11) along the parabolic

segment.

2.2 A variational formulation in the hyperbolic region.

The smooth solution in the elliptic region provides all the data on the parabolic segment which is needed to solve either a Goursat problem or a Cauchy problem in the characteristic triangle bounded by $\Gamma_1 \cup \Gamma_2$ and the parabolic basis.

(1) Consider the Goursat problem:

$$yu_x - v_y = 0 \quad \text{in} \quad \Omega_2 \tag{12a}$$

$$u_y - v_x = 0$$

with the initial conditions:

$$u(x,o) = u_o(x) \qquad\qquad -1 \le x \le 1$$

and either $u \sqrt{y} + v \mid_{\Gamma_1} = f(y)$ or an equivalent condition for λ: (12b)

$$\lambda \mid_{\Gamma_1} = 2 \int \sqrt{y} \, f(y) \, dy$$

The variational formulation for the problem is:

Find $(u,v;\lambda) \in U \times V \times W$ satisfying (12b) **for λ having fixed** (unknown) values along Γ_2, for which the functional:

$$J(u,v;\lambda) = \int\int_{\Omega_2} [yu^2 - v^2 + \lambda(u_y - v_x)]dxdy +$$

$$\int_{\Gamma_1 \cup \Gamma_2} \lambda(udx + vdy) \tag{12c}$$

is stationary.

This variational formulation in the hyperbolic region is a con-
tinuation of the saddle-point formulation used in the elliptic region.
But, in the hyperbolic domain $(y > 0)$ conditions (8a), (8b) are not
satisfied, and we loose the proof for the existence and the uniqueness of
the solution of the variational problem.

(ii) Cauchy problem for the Tricomi equation.

The given boundary conditions for the problem are:

$$u(x,0) = u_0(x)$$

$$-1 \leqslant x \leqslant 1 \qquad (13a)$$

$$v(x,0) = v_0(x)$$

An equivalent variational formulation [13 chapter 3] is:

Find $(u,v;\lambda)$ such that $(u,v) \in U \times V$ and satisfying (13a)
and $\lambda \in W$ with fixed (unknown) values along $\Gamma_1 \cup \Gamma_2$ for which
the functional:

$$J(u,v;\lambda) = \int_{\Omega_2} \int [yu^2 - v^2 + \lambda(u_y - v_x)]dxdy +$$

$$\int_{\Gamma_1 \cup \Gamma_2} \lambda(udx + vdy) \qquad (13b)$$

is stationary.

3. EXTREMAL PROPERTIES OF THE VARIATIONAL FUNCTIONAL

Conditions (8a), (8b) given by Brezzi [9] are satisfied only in the elliptic region. These conditions are connected with the following:

Given

$$J(u,v;\lambda) = \int_{\Omega_1} \int [yu^2 - v^2 + \lambda(u_y - v_x)] dxdy$$

for any given λ, the extremum of the functional, noted by $(u_0(\lambda),\ v_0(\lambda))$ satisfies the equations:

$$2yu - \lambda_y = 0$$

$$2v - \lambda_x = 0$$

hence $\qquad yu_x - v_y = 0.$

For hyperbolic and mixed type equations $(u_0(\lambda),\ v_0(\lambda))$ is a saddle point which may be specified in the hyperbolic case in what follows: The projection of the functional on the U space   and on the V space separately, yields an extremum in each of these cases; the projection on the U space   gives:

$$\delta^2 J(u^\circ(v,\lambda))\Big|_U = \int_{\Omega} \int 2y\ dxdy\ ,\quad \text{for}\ \ y > 0,$$

so $u^{\circ}(v,\lambda)$ is a minimum in U, and for the space V, the second variation of $J(u,v\;;\lambda)\Big|_V$ is

$$\delta^2 J(v^{\bullet}(u,\lambda))\Big|_V = -\int_\Omega\!\!\int 2dxdy$$

so $v^{\circ}(u,\lambda)$ is a maximum in V.

Hence, for the hyperbolic region the "point"

$$(u^{\circ}(\lambda),v^{\circ}(\lambda)) \text{ is a min max } J(u,v;\lambda).$$
$$u\in U \;\; v\in V$$

For mixed type equations the situation is more complicated.

4. APPROXIMATION

We use the variational principle (4) to solve, approximately, the Tricomi equation in a mixed domain.

The boundary conditions for the 4 examples solved are chosen so that the exact solution for the potential $\phi(x,y)$ is:

$$\text{I.} \qquad \phi(x,y) = \cosh x \left(y + \sum_{n=1}^{\infty} \frac{y^{3n+1}}{(3n+1)3n(3n-2)\cdots 4\cdot 3} \right)$$

$$\text{II.} \qquad \phi(x,y) = 3x^2 + y^3$$

$$\text{III.} \qquad \phi(x,y) = x^3 + xy^3$$

$$\text{IV.} \qquad \phi(x,y) = 2x^3 y + xy^4.$$

The numerical results are illustrated in Tables (I-V) for the first example (the values of the rate of convergence of the other examples behave almost the same).

i) The Dirichlet problem for the Tricomi equation in the elliptic domain. Given the problem:

$$y\phi_{xx} - \phi_{yy} = 0$$

$$\phi\Big|_{\partial\Omega} = f(x,y)$$

The variational functional is:

$$J(u,v;\lambda) = \int_{\Omega}\int [yu^2 - v^2 + (u_y - v_x]dxdy$$

for $(u,v) \in U \times V$, $\lambda \in W$

$$U \times V = \{(u,v) / \int_{\Omega} \int [-yu^2 + v^2 + (u_y - v_x)^2] dx dy < \infty,$$

$$udx + vdy \Big|_{\partial \Omega} = df\}$$

$$W = \{q / \int_{\Omega} \int q^2 dx dy < \infty , \int_{\Omega} \int q dx dy = o\} ,$$

for this problem conditions (8a), (8b) are fulfilled, hence the problem has a unique solution.

Let Ω be the rectangle:

$$\Omega = \{(x,y) / -1 \leq x \leq 1 , -1 \leq y \leq 0\}$$

we divide Ω into rectangles of size h and take different finite dimensional space for the trial functions.

a) The first attempt is to use bilinear trial functions for for (u,v) and λ, for these space Brezzi's conditions (9a), (9b) and (9c) are not satisfied; this is concluded from the instability of the numerical solution. For these trial functions, central approximation for λ_x and λ_y are received and therefore the value of λ at one corner determines its' values only at every second point in each direction. To couple the equations for

all grid points, the values of λ at the 4 points of the corner element have to be predetermined (e.g. by Taylor's expansion about the $(-1,-1)$ point and the values of u,v along the boundary).

The L_2 error of the functions u,v,λ for the 4 examples given above is approximately $O(h^2)$ for the Dirichlet problem (**see Table Ia**).

The experimental rate of convergence is reduced to 1.6 if the projection (11) is used along the parabolic segment. (see Table IIIa).

b) Other finite dimensional spaces for U_h, V_h and W_h for which (9a), (9b) and (9c) are satisfied, and tried (Appendix A). Let us use the same finite elements discretization of Ω as in a), approximating u and v by bilinear trial functions but piece-wise constant functions for λ (intuitively, this leaves us with the only arbitrary constant of the problem which is $\int_\Omega \int q \, dx \, dy = 0$).

For this approximation the solution converges to the analytic solution for both of the problems; The Dirichlet problem and problem with the integral condition (11) (without adding artificial conditions for stability).

The rate of convergence of the L_2 error for u and v is about 1.4 and 1 for λ (see Tables Ib and IIIb).

ii) Goursat conditions for the hyperbolic region.

The smooth solution in the elliptic part provides the values of u on the parabolic line $A\ B$ $(y = 0, -1 \leqslant x \leqslant 1)$, which, together with given boundary conditions along the characteristic Γ_1 provides the data needed for the Goursat problem in the triangle ABC (well posed for the continuous case). The region is now divided into isoparametric triangles formed by the characteristic mesh in $\hat{\Omega}_2$ (figure 2). Linear trial functions in ξ and η (the characteristic variables) are used:

$$\nu_m^e = a_m^e \eta + b_m^e \xi + c_m^e$$

where

$$\nu_m^e = (u^e, v^e, \lambda^e)$$

in the triangular isoparametric elements.

The scheme is inconsistent near the parabolic line. To overcome this difficulty, linear elements in x,y coordinates and linear trial functions are taken for the first row of elements.

The functional J_2 (12c) is discretized and the nodal values (u^e, v^e, λ^e) sought from the discrete algebraic conditions for stationarity.

The numerical solution of the same examples used for the elliptic region for accurate boundary conditions is convergent and the experimental rate of convergence of the L_2 error is about 1.5 (see Table II).

(iii) Cauchy conditions for the hyperbolic region Ω_2.

An alternative way to treat the hyperbolic region is to consider $(u,v,\lambda)(x,0)$ on AB from the elliptic solution as Cauchy (or initial) data for the hyperbolic triangle ABC, and use the functional (13b) for each isoparametric triangular element separately. This procedure also yields a well posed problem with explicit algebraic linear equations.

The solution of this approximation has a smaller L_2 error than the solution for the Goursat problem, the rate of convergence of the L_2 error is about 1.7 (see Table II).

Note: All implicit schemes for the Cauchy problem seem to be unstable and explicit schemes with Goursat boundary conditions do not converge.

(iv) Approximation for the mixed domain.

As has been shown in the previous chapter, a solution for the mixed domain is available.

The first step is to solve the problem in the elliptic region using the integral projection (11). The values of u and v and λ along the parabolic segment are achieved by this calculation and may be used as data in order to get a solution in the hyperbolic region by solving either a Goursat problem or a Cauchy problem. The results are illustrated in Tables IIIa and IIIb.

If λ is bilinear in the elliptic region, we get large errors in the values of λ, especially in the last example. On the parabolic line λ is used as initial conditions for the Cauchy problem but not for the Goursat problem. Therefore, the rate of convergence is better using Goursat boundary conditions.

On the other hand, if piecewise constant function is taken for λ in the elliptic region, the Cauchy conditions give better results than the Goursat conditions. The schemes for the Cauchy problem are explicit; hence, the procedure is better, and finer meshes for the hyperbolic region are available.

The disadvantage of the process described before is that the integral of Bitsadze (11) needs much time and must be recomputed for different boundary conditions.

In examples II and III difficulties in the convergence of the integral

$$\int_0^x \frac{\psi(t/2)}{t^{2/3}(x-t)^{1/6}}\, dt$$

occurred especially for small values of x. We can overcome this problem by adding a constant to $\Phi(x,y)$ so that the new function is zero at $x = -1$. For a general function integration by parts must be done, which costs more computation time.

Another disadvantage of the Bitsadze relation (11) is that it can be used only for the Tricomi equation. For another mixed problem another projection formula has to be found; therefore, we look for

a numerical process which connects between the hyperbolic and the elliptic region.

The hyperbolic region Ω_2 is divided into isoparametric triangular elements as described before (Figure 2). We define a local variational formulation for each element $E^{i,j}$ separately:

$$J^{i,j}(u,v,\lambda) = \int_{E^{i,j}} \int [yu^2-v^2+\lambda(u_y-v_x)] \, dxdy +$$

$$\int_{S_1^{i,j} \cup S_2^{i,j}} \lambda(udx + vdy)$$

where $S_1^{i,j}$ and $S_2^{i,j}$ are the two characteristics bounding the isoparametric element, and use linear trial functions as before.

If $U^{(0)}$ - the solution on $y = 0$ is not known, the explicit scheme enables us to find a connection between $U^{(0)}$ and the boundary condition along Γ_1. For every j we can find a matrix B^j, so $U^{(j)} = B^{(j)}U^{(0)}$ where $U^{(j)}$ is the solution at the j-th row, and B^j is a matrix depending only on the size of the mesh and not on the boundary conditions.

Substituting the boundary conditions along Γ_1, we get one equation for every point on the parabolic line. The two other equations for the three unknowns (u,v,λ) will come from the schemes for the elliptic region. After solving the elliptic problem with the numerical projection, we get immediately the solution in the hyperbolic region using the explicit scheme.

In the solution of the elliptic region oscillations of the values of u along the parabolic segment are observed. The process converges in spite of the oscillations. In order to demonstrate this phenomenon, the error of u is calculated in the usual L_2 norm and in the weighted norm: $\sqrt{\int_\Omega \int yu^2 dxdy}$ (Table IV).

The oscillations can be removed by averaging the values of u along the parabolic segment, and using the smoothed values as Dirichlet conditions for the problem, the L_2 error of u is reduced and the L_2 error of vv and λ is unchanged (Table V).

For λ bilinear in $y \leq 0$, the rate of convergence in the elliptic region is about 1.3 for the first example but only 0.5 in the last example; contrary to the case where λ is piecewise constant, there the rate of convergence is the same for all boundary conditions considered. Therefore, it is not worthwhile fo use the numerical projection together with λ bilinear in the elliptic region (Appendix B).

Good results with λ bilinear in the elliptic region are achieved only by using the integral relation (11) and the implicit schemes of the Goursat problem in the hyperbolic region; hence, much computation time is needed in this case. Because of all these disadvantages, it is preferable to use λ piecewise constant for all the problems.

5. SUMMARY, CONCLUSIONS AND REMARKS

Tricomi boundary value problem in a mixed domain is solved on a
finite-element mesh, using a variational formulation admitting arbit-
rary initial and/or boundary conditions. Convergent solutions are ob-
tained, by first solving the elliptic part using either an integral condi-
tion or a numerical projection of boundary condition along the para-
bolic line, then extending the solution into the hyperbolic part on
characteristic finite elements, solving either a Goursat boundary val-
ue problem, or a Cauchy initial one.

The advantage of this approach over others (e.g. the one using
Friedrich's ingenious, yet non-trivial procedure from the practical
point of view even for the simplest cases) is that it is conceptually
simple, does not require further transformations, and can be extended
to nonlinear problems and to multi-dimensional and unsteady situations,
at least in principle.

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Appendix A: Stability proof for the elliptic region.

To prove the stability of the numerical procedure for the elliptic region $y < 0$, using bilinear approximation for u and v and piecewise constants for λ, the following condition has to hold [9]:

$$(14) \qquad \sup_{(u,v)\in U_h \times V_h} \frac{\iint\limits_{\Omega} \Psi(u_y - v_x)dxdy}{||u,v||} \geq \ell \, ||\Psi||$$

for every $\Psi \in W_h$

where

$U_h \times V_h = \{(u,v)/ \ u$ and v are bilinear functions in each rectangular element, and

$$\iint\limits_{\Omega}[-yu^2 + v^2 + (u_y - v_x)^2]dxdy < \infty, \quad udx + vdy_{|\partial\Omega} = 0\}$$

with the norm

$$||(u,v)||^2_{U_h \times V_h} = \iint\limits_{\Omega}[-yu^2 + u^2 + (u_y - v_x)^2]dxdy,$$

and

$W_h = \{\Psi|\Psi$ is piecewise constant in each rectangle

and $\quad \iint\limits_{\Omega}\Psi^2 dxdy < \infty, \quad \iint\limits_{\Omega}\Psi dxdy = 0\}$

with the norm

$$||\Psi||^2 = \iint\limits_{\Omega}\Psi^2 dxdy.$$

Consider the patch of four elements shown in figure 3 (the same holds for any number of elements).

Let us show that for each element E_i

where $\quad\quad\quad \Psi = \Psi_i = \text{constant}$

and $\quad\quad\quad \sum_i \Psi_i = 0$

it is possible to find a pair of functions $(u,v) \in U_h \times V_h$ for which

$$(15) \quad\quad \iint\limits_{E_i} (u_y - v_x)\, dxdy = \Psi_i \quad\quad \bullet$$

The algebric system of (15) for (u_i, v_j) $i = 0, 1, \ldots 8$ is:

$$\frac{-(u_0 + u_1)}{2k} - \frac{v_0 + v_3}{2h} = \Psi_1$$

$$\frac{-(u_0 + u_5)}{2K} + \frac{v_3 + v_0}{2h} = \Psi_2$$

$$\frac{u_0 + u_5}{2k} + \frac{v_0 + v_7}{2h} = \Psi_3$$

$$\frac{u_0 + u_1}{2k} - \frac{v_0 + v_7}{2h} = \Psi_4$$

and

$$\sum_{i=1}^{4} \Psi_i = 0 \ ,$$

and the solution is:

$$
\begin{pmatrix} \dfrac{u_o + u_1}{2k} \\[2em] \dfrac{u_o + u_5}{2k} \\[2em] \dfrac{v_o + v_3}{2h} \\[2em] \dfrac{v_o + v_7}{2h} \end{pmatrix} = \begin{pmatrix} \Psi_3 + \Psi_4 \\[2em] 0 \\[2em] \Psi_2 \\[2em] \Psi_3 \end{pmatrix} + t \begin{pmatrix} 1 \\[2em] -1 \\[2em] -1 \\[2em] 1 \end{pmatrix}
$$

Let us choose $u_o = v_o = 0$, $t = 0$, then

$$
\iint\limits_{E_i} (u_y - v_x)^2 \, dxdy = \sum_{i,j=1}^{4} r_{ij} (\Psi_i - \Psi_j)^2 \le \alpha \sum_{i=1}^{4} \Psi_i^2
$$

If the region is devided into more elememts, the constant α is not changed, hence α is independent on the mesh.

Calculating $||(u,v)||^2$ we obtain:

$$
\iint\limits_{\Omega} [-yu^2 + v^2 + (u_y - v_x)^2] \, dxdy \le \gamma \iint\limits_{\Omega} [u^2 + v^2 + (u_y - v_x)^2] \, dxdy
$$

$$
\le (\delta + \epsilon h^2 + \eta k^2) hk \sum \Psi_i^2 \qquad .
$$

Since $h, k \le D$, where D depends on the region only,

$$
\iint\limits_{\Omega} [-yu^2 + v^2 + (u_y - v_x)^2] \, dxdy \le L^2 hk \sum \Psi_i^2
$$

L   is independent of h and k, unless they are of the same order for any

patch of rectangular elements, and condition (15) is proved, since:

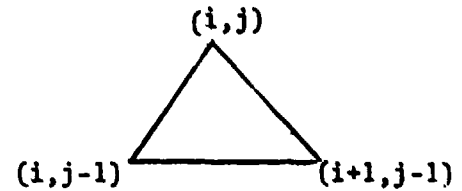$$\sup_{(u,v) \in U_n \times V_h} \frac{\iint_\Omega \Psi(u_y - v_x) dx dy}{||(u,v)||} \geq \frac{1}{L} ||\Psi||$$

for every $\Psi \in W_h$.

Appendix B: Oscillations of u on the parabolic line using numerical projection

1) Possible reason for the numerical oscillations

The explicit schemes in the hyperbolic region with Cauchy boundary conditions, using functions which are linear in x, y, are:

(1a)
$$\lambda_{i,j} = \frac{\lambda_{i,j-1} + \lambda_{i+1,j-1}}{2}$$

$$+ \left(\frac{3}{4}h\right)^{2/3}\left[j^{2/3} - (j-1)^{2/3}\right]\frac{1}{10}\left[u_{i,j}\left(6j^{2/3} + 4(j-1)^{2/3}\right)\right.$$

$$+ \left.\left[u_{i,j-1} + u_{i+1,j-1}\right]\left(2j^{2/3} + 3(j-1)^{2/3}\right)\right]$$

(1b)
$$v_{i,j} = -0.5\left[v_{i,j-1} + v_{i+1,j-1}\right] + \frac{1}{h}\left[\lambda_{i+1,j-1} - \lambda_{i,j-1}\right]$$

(1c)
$$u_{i,j} = 0.5\left[u_{i,j-1} + u_{i+1,j-1}\right] + \frac{\left(\frac{3}{4}\right)^{2/3}}{h^{1/3}}\left[j^{2/3} - (j-1)^{2/3}\right]\left[v_{i+1,j-1} - v_{i,j-1}\right]$$

$u_{i,j-1}$ and $u_{i+1,j-1}$ have the same coefficient and the same sign in all the schemes. Therefore $u_{i,0}-d$ and $u_{i+1,0}+d$ lead to the same solution in the hyperbolic region. The same consideration cannot be done for v and λ because they have different signs in some of the schemes.

The numerical projection is obtained by substituing the boundary conditions along Γ_1 (figure 2):

$$u_{-N,j}\sqrt{y} + v_{-N,j} = f(y)$$

where $u_{-N,j}$ is obtained from (1c) and $v_{-N,1}$ from (1b).

For the first step, $j = 1$, and $y = (\frac{3}{4}h)^{2/3}$, we project the condition

$u_{-N,1}(\frac{3}{4}h)^{1/3} + v_{-N,1} = f(y)$ on the segment $\{-Nh \leq x \leq (-N+1)h, y = 0\}$

The values of $u_{-N,0}$ and $\lambda_{-N,0}$ are known $(v_{-N,0} = f(0)$ and $u_{-N,0}$ is computed

by L' Hopital's rule and λ is taken as an arbitrary value at this point), therefore

the first projection gives a connection between $u_{-N+1,0}$, $v_{-N+1,0}$ and $\lambda_{-N+1,0}$:

(2) $$\frac{u_{-N+1,0}}{2} \left(\frac{3}{4}h\right)^{1/3} + \frac{v_{-N+1,0}}{4} + \frac{1}{h}\lambda_{-N+1,0} = \text{const} .$$

The coefficient of $u_{-N+1,0}$ decreases to zero when $h \to 0$, contrary to the

coefficients of $v_{-N+1,0}$ and $\lambda_{-N+1,0}$. For a final value of h equation (2)

enables a large error of $u_{-N+1,0}$ and this error may increase if finer meshes are

used.

Assume that $u_{-N+1,0} = u_{-N+1,0}^{(anal)} + d$. As we have seen before we can add $\pm d$ to

$u_{i,0}$ and $u_{i+1,0}$ accordingly, without changing the solution in the hyperbolic region.

Therefore oscillations occure in the solution of u on $y = 0$.

2. Explanation why good results are achieved with λ piecewise constant but bad convergence using λ bilinear in the elliptic region
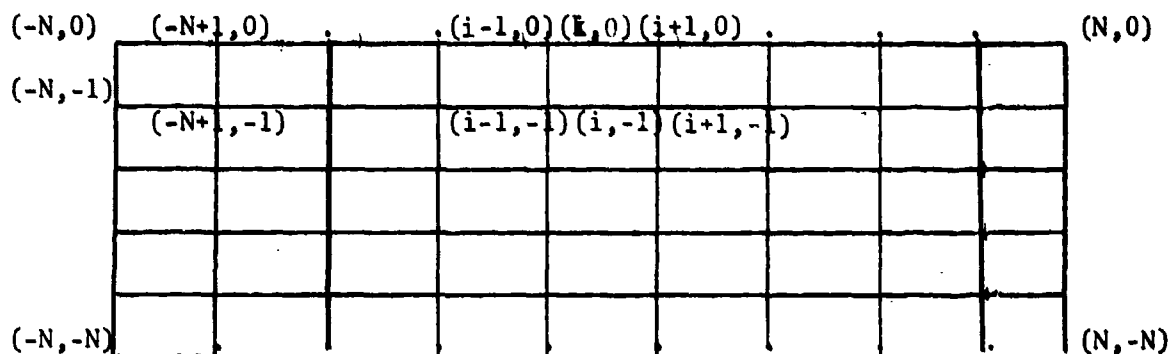
The boundary conditions for the points on the parabolic line are:

i) the numerical projection

ii) $\dfrac{\partial J}{\partial v_{i,0}} (u,v,\lambda) = 0$

iii) $\dfrac{\partial J}{\partial \lambda_{i,0}} (u,v,\lambda) = 0$

where J is the functional in the elliptic region:

The values of $u_{i,0}$ appear in the numerical projection and in the schemes

$\dfrac{\partial J}{\partial u_{i,-1}} = 0$ and $\dfrac{\partial J}{\partial \lambda_{i,0}} = 0$ (and $\dfrac{\partial J}{\partial \lambda_{i,-1}}$ if λ is bilinear). $\dfrac{\partial J}{\partial u_{i,-1}} = 0$

gives the following coefficients:

$$-h \, (4u_{i,0} + u_{i+1,0} + u_{i-1,0}) + \cdots$$

Therefore, if there are oscillations, they are multiplied by $\tfrac{1}{2}h$ and do not influence the results for interior points for small h. On the other hand, $\dfrac{\partial J}{\partial \lambda_{i,0}} = 0 \Rightarrow$

(a) λ bilinear

$$4 (u_{i,0} - u_{i,-1}) + (u_{i+1,0} - u_{i-1,-1}) + (u_{i-1,0} - u_{i-1,-1}) =$$

$$2 (v_{i+1,0} - v_{i-1,0}) + (v_{i+1,-1} - v_{i-1,-1})$$

(b) λ piecewise constant

$$u_{i,0} - u_{i,-1} + u_{i+1,0} - u_{i+1,-1} = v_{i+1} - v_{i,0} + v_{i+1,-1} - v_{i,-1}$$

In case (a) $u_{i-1,0}, u_{i,0}$ and $u_{i+1,0}$ have different coefficients.
Therefore, oscillations on the parabolic line will disturb the results
in the elliptic region.

In case (b) $u_{i,0}$ and $u_{i+1,0}$ have the same coefficient; hence, the oscil-
lations along the parabolic line do not affect the other variables inside the
region.

3. Numerical results

If λ is piecewise constant, the oscillations on $y = 0$ can be
decreased by fixing the arbitrary value of λ at the point $(N,0)$
instead of $(-N,0)$. Then the equation for the point $(-N,0)$ is:

$$\frac{\partial J}{\partial \lambda}_{-N,0} = 0 \Rightarrow u_{-N,0} - u_{-N,-1} + u_{-N+1,0} - u_{-N+1,-1}$$

$$= v_{-N+1,0} - v_{-N,0} + v_{-N+1,-1} - v_{-N,-1}.$$

We know the values of $u_{-N,0}, v_{-N,0}, v_{-N,-1}$ from the boundary conditions,
so we have the following connection:

$$(3) \qquad u_{-N+1,0} - u_{-N+1,-1} - u_{-N,-1} - v_{-N+1,0} - v_{-N+1,-1} = 0$$

In this scheme the coefficient of $u_{-N+1,0}$ does not decrease to zero
when $h \to 0$ in contrary to (2). Therefore, the error d will be smaller.

Notes:

1. In the elliptic region with Dirichlet conditions and λ piecewise
    constant, we do not use the value of $\lambda_{N,0}$ because we take in every
    rectangle the value of λ at the left and upper corner. But if
    the elliptic problem is solved with the numerical projection, then
    there is a scheme, belonging to the hyperbolic part, which contains
    $u_{N,0}$, $v_{N,0}$ and $\lambda_{N,0}$. In this case we fix the value of $\lambda_{N,0}$. If $\lambda_{-N,0}$
    is given as an arbitrary value, we must add an equation for $\lambda_{N,0}$.
    We take, for example, extrapolation of $\lambda_{N-1,0}$ and $\lambda_{N-2,0}$. But, as
    we have explained before, larger oscillations appear in this case.

2. If λ is bilinear in the elliptic region, 4 values of λ are given
    around the corner $(-N, -N)$ in both cases; the Dirichlet problem
    and the problem with the numerical projection.

    In Table VI the results are illustrated for the first example.
    In this case $u_{anal}(x,0) = 0$. One can see that the size of the
    oscillations do not depend on h but they increase when x approaches 1.
    The solution does not converge locally on $y = 0$ but converges in
    L_2 - norm and in the weighted norm $(\iint \mu u^2 dx dy)^{1/2}$ (Table V).

4. <u>Correction on the parabolic line</u>

    We compute

    $$di = \frac{u_{i-1,0} - 2u_{i,0} + u_{i+1,0}}{4} \qquad \forall i \quad -N < i < N$$

    where $u_{i,0}$ is the solution on $y = 0$ of the elliptic problem with the
    numerical projection.

We define new boundary conditions on the parabolic line:

$$\overline{u}_{-N,0} = u_{-N,0} - d_{-N+1}$$

$$\overline{u}_{i,0} = u_{i,0} - d_i$$

$$\overline{u}_{N,0} = u_{N,0} - d_{N-1}$$

In the first example we have:

$$0.68 \leq |d_i| \leq 0.98 \qquad\qquad h = 1/4$$

$$0.49 \leq |d_i| \leq 1.03 \qquad\qquad h = 1/8$$

$$0.33 \leq |d_i| \leq 1.01 \qquad\qquad h = 1/16$$

and the smoothed values $|\overline{u}_{i,0}| \leq 0.06$.

We solve the elliptic region again but with $\overline{u}_{i,0}$ as Dirichlet boundary conditions on $y = 0$. $\lambda_{N,0}$ does not appear in the Dirichlet problem if λ is piecewise constant; therefore, $\lambda_{N-1,1}$ from the previous solution is taken as an arbitrary value for λ. The results are given in Table V.

<u>Legend</u>

$$EU = \left(\sum (u_{num} - u_{anal})^2 \; \Delta x \Delta y \right)^{1/2}$$

$$EUy = \left(\sum y(u_{num} - u_{anal})^2 \; \Delta x \Delta y \right)^{1/2}$$

$$EV = \left(\sum (v_{num} - v_{anal})^2 \; \Delta x \Delta y \right)^{1/2}$$

$$EG = \left(\sum (\lambda_{num} - \lambda\lambda_{anal})^2 \; \Delta x \Delta y \right)^{1/2}$$

	$h = 1/4$	$h = 1/8$	$h = 1/16$	Rate of Convergence
EU	0.0107	0.0025	0.0006	2.15
EV	0.0453	0.0104	0.0025	2.09
EG	0.2270	0.0549	0.0135	2.05

Table 1a: Results of the errors and rates of convergence for the Dirichlet problem using λ bilinear.

	$h = 1/4$	$h = 1/8$	$h = 1/16$	Rate of Convergence
EU	0.0629	0.0257	0.0100	1.33
EV	0.0962	0.0362	0.0133	1.43
EG	0.334	0.184	0.095	0.91

Table 1b: Results of the errors and rates of convergence for the Dirichlet problem using λ piecewise constant.

	h = 1/4	h = 1/8	h = 1/16	Rate of Convergence
EU:				
Goursat	0.116	0.045	0.017	1.38
Cauchy	0.00289	0.00092	0.00033	1.63
EUy:				
Goursat	0.091	0.034	0.012	1.53
Cauchy	0.00269	0.00086	0.00028	1.63
EV:				
Goursat	0.098	0.035	0.012	1.52
Cauchy	0.0338	0.0093	0.0028	1.77
EG:				
Goursat	0.006	0.0022	0.0006	1.69
Cauchy	0.026	0.0093	0.0033	1.49

Table II: Results of the errors and rates of convergence in the hyperbolic domain with exact boundary conditions for either the Goursat or Cauchy problem.

	h = 1/4	h = 1/8	h = 1/16	Rate of Convergence
EU:				
$y \leq 0$	0.265	0.079	0.020	1.86
$y > 0$, Goursat	0.262	0.083	0.027	1.64
$y > 0$, Cauchy	0.160	0.111	0.082	0.48
EUy:				
$y \leq 0$	0.178	0.059	0.016	1.73
$y > 0$, Goursat	0.212	0.064	0.020	1.70
$y > 0$, Cauchy	0.138	0.086	0.059	0.61
EV:				
$y \leq 0$	0.176	0.057	0.015	1.88
$y > 0$, Goursat	0.269	0.076	0.022	1.80
$y > 0$, Cauchy	0.169	0.095	0.062	0.71
EG:				
$y \leq 0$	0.634	0.233	0.067	1.62
$y > 0$, Goursat	0.534	0.202	0.058	1.60
$y > 0$, Cauchy	0.127	0.048	0.012	1.70

Table IIIa: Results of the errors and rates of convergence in the
mixed domain using Bitsadze relation (11).

$y \leq 0$ λ bilinear

$y > 0$ Goursat or Cauchy

	$h = 1/4$	$h = 1/8$	$h = 1/16$	Rate of Convergence
EU:				
$y \leq 0$	0.183	0.066	0.018	1.67
$y > 0$, Goursat	0.339	0.160	0.054	1.33
$y > 0$, Cauchy	0.176	0.071	0.024	1.44
EUy:				
$y \leq 0$	0.071	0.027	0.009	1.49
$y > 0$, Goursat	0.285	0.129	0.042	1.54
$y > 0$, Cauchy	0.161	0.064	0.022	1.44
EV:				
$y \leq 0$	0.119	0.042	0.014	1.55
$y > 0$, Goursat	0.335	0.140	0.044	1.47
$y > 0$, Cauchy	0.239	0.086	0.028	1.55
EG:				
$y \leq 0$	0.452	0.223	0.103	1.07
$y > 0$, Goursat	0.361	0.131	0.033	1.23
$y > 0$, Cauchy	0.059	0.022	0.008	1.44

Table IIIb: Results of the errors and rates of convergence in the mixed domain using Bitsadze relation (11).

$y \leq 0$: λ piecewise constant

$y > 0$ Goursat or Cauchy

	h = 1/4	h = 1/8	h = 1/16	Rate of Convergence
EU:				
$y \leq 0$	0.611	0.426	0.288	0.54
$y > 0$	0.224	0.098	0.041	1.23
EUy:				
$y \leq 0$	0.081	0.038	0.018	1.08
$y > 0$	0.172	0.070	0.028	1.31
EV:				
$y \leq 0$	0.143	0.060	0.026	1.23
$y > 0$	0.172	0.070	0.028	1.31
EG:				
$y \leq 0$	0.374	0.174	0.083	1.08
$y > 0$	0.123	0.038	0.012	1.68

Table IV: Results of the errors and rates of convergence in the mixed domain using numerical projection.

$y \leq 0$: λ piecewise constant

$y > 0$: Cauchy

	h = 1/4	h = 1/8	h = 1/16	Rate of Convergence
EU:				
$y \leq 0$	0.133	0.071	0.037	0.92
$y > 0$	0.053	0.029	0.015	0.91
EUy:				
$y \leq 0$	0.076	0.036	0.016	1.13
$y > 0$	0.050	0.026	0.012	1.03
EV:				
$y \leq 0$	0.124	0.052	0.022	1.25
$y > 0$	0.192	0.068	0.024	1.50
EG:				
$y \leq 0$	0.262	0.123	0.061	1.05
$y > 0$	0.035	0.009	0.002	2.00

Table V: Results of the errors and rate of convergence in the mixed domain, using numerical projection and smoothing of the solution on the parabolic line.

$y \leq 0$: λ piecewise constant

$y > 0$ Cauchy

$x \backslash u_{num}$	$h = 1/4$	$h = 1/8$	$h = 1/16$
- 1	0	0	0
			0.34
- 7/8		0.48	-0.27
			0.42
- 3/4	0.65	-0.43	-0.41
			0.50
- 5/8		0.59	-0.52
			0.57
- 1/2	-0.65	-0.64	-0.61
			0.64
- 3/8		0.70	-0.69
			0.70
- 1/4	0.75	-0.80	-0.75
			0.75
- 1/8		0.79	-0.81
			0.80
0.	-0.94	-0.91	-0.86
			0.83
1/8		0.86	-0.90
			0.87
1/4	0.84	-1.00	-0.94
			0.90
3/8		0.92	-0.97
			0.92
1/2	-1.10	-1.06	-1.00
			0.94
5/8		0.95	-1.01
			0.96
3/4	0.88	-1.09	-1.03
			0.98
7/8		1.01	-1.03
			1.03
1.	-0.82	-0.78	-0.74

$u_{anal} (x,0) = 0$

Table VI: The results of u on y = 0 solving the elliptic problem with the numerical projection and using λ piecewise constant.
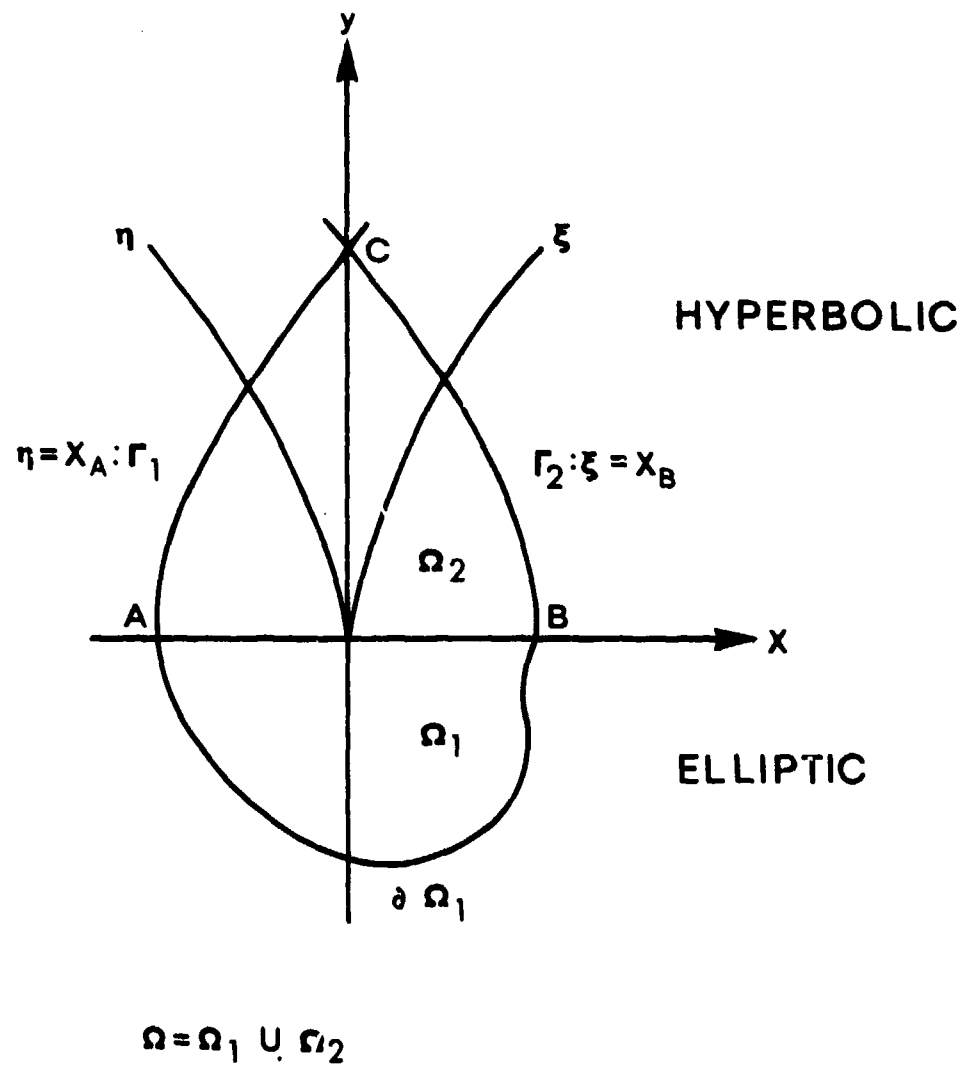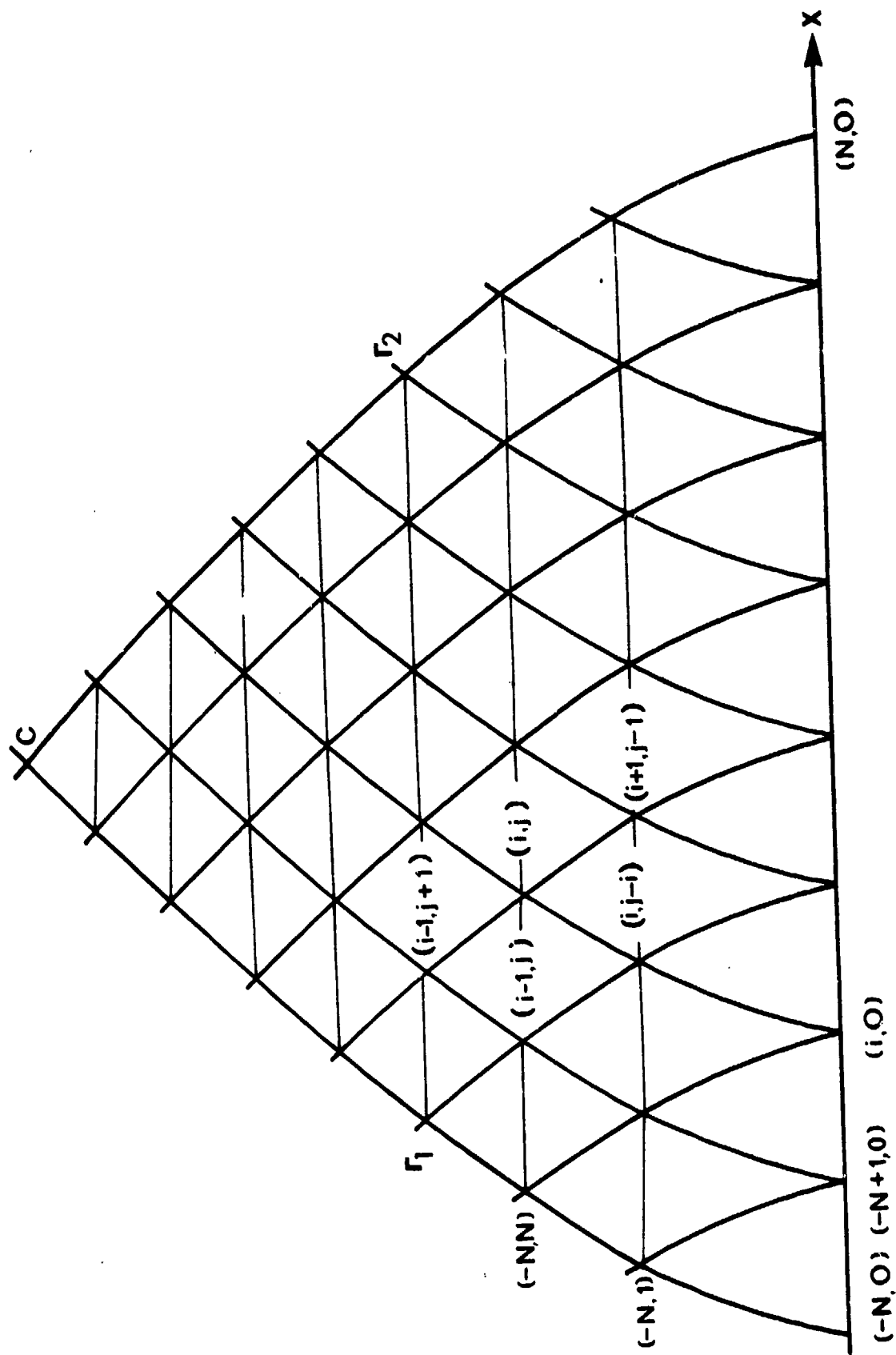
Figure 1: Tricomi Domain.

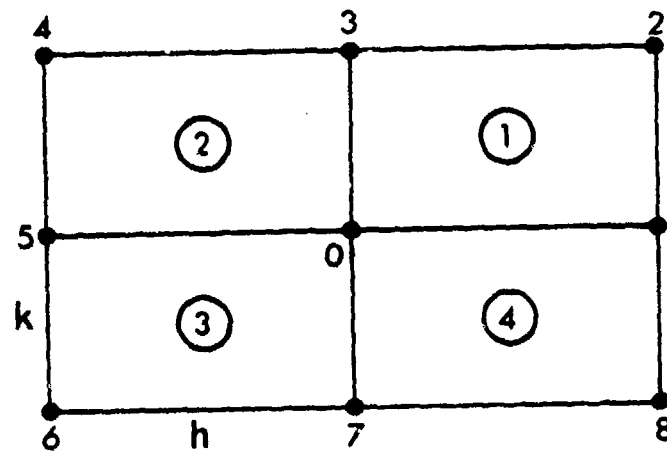Figure 2: Characteristic Elements in Hyperbolic Region.

Figure 3: A patch of elements.

UNIQUENESS OF THE SOLUTION OF THE LAPLACE EQUATION

WITH SPECIAL NON-LINEAR BOUNDARY CONDITIONS

Sara Yaniv

Abstract

Non-standard, non-linear boundary conditions for the Laplace
equation are considered, and the question of uniqueness examined.
The nonlinear conditions prescribe the magnitude of the gradient
on part of the boundary, and with either Dirichlet or Newmann con-
ditions prescribed on the remaining part. The solution of the
problem is unique up to a sign.

Non-linear boundary value problem

Consider Laplace equation:

$$\Delta U = 0 \tag{1a}$$

in a two dimensional domain Ω, bounded by

$$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$$

with the non-standard boundary conditions:

$$\left. U \right|_{\partial\Omega_1} = 0 \tag{1b}$$

$$\left. |\nabla U| \right|_{\partial\Omega_2} = 1$$

The solution for problem (1) is not unique; the question is how many solutions there are and what is the connection among them.

Theorem 1:

If $U_1(x,y)$ is a solution of problem (1a), (1b), then $U_1(x,y)$ and $-U_1(x,y)$ are the only solutions of the problem.

Proof:

Using complex functions of a complex variable $z = x + iy$ we define the function

$$f(z) = \phi(x,y) + i\psi(x,y),$$

$f'(z)$ is an analytic function satisfying the following **boundary conditions**:

$$\text{Re } f\Big|_{\partial\Omega_1} = 0 \qquad\qquad (2)$$

or

$$\text{Re } df\Big|_{\partial\Omega_1} = \text{Re } f' \, dz\Big|_{\partial\Omega_1} = 0 \qquad\qquad (2')$$

(which is equivalent to $dU = U_x \, dx + U_y \, dy\Big|_{\partial\Omega_1} = 0$)

and
$$\left| f' \right|_{\partial\Omega_2} = \left| \frac{df}{dz} \right|_{\partial\Omega_2} = 1 \qquad\qquad (3).$$

Let us test the problem for the analytic function $f'(z) = u(x,y) - iv(x,y)$ satisfying conditons $(2')$ and (3) .

Assuming f'_1 and f'_2 are analytic functions satisfying $(2')$ and (3), then

$$f'_2\Big|_{\partial\Omega_2} = f'_1 e^{i\alpha(z)}\Big|_{\partial\Omega_2}$$

and
$$\text{Im } \alpha(z)\Big|_{\partial\Omega_2} = 0$$

since f_2' is an analytic function, we write:

$$f_2' = f_1' e^{i\alpha(z)} \quad \text{in} \quad \Omega$$

where

$$\alpha(z) = a(x,y) + ib(x,y)$$

is also a analytic function satisfying:

$$b(x,y)\Big|_{\partial\Omega_2} = 0. \tag{4}$$

Condition (2') leads to the following relation between f_1' and f_2' :

$$\text{Re} f_2' \, dz\Big|_{\partial\Omega_1} = \text{Re} \, f_1' \, e^{i\alpha(z)}\Big|_{\partial\Omega_1} = 0$$

or

$$\text{Re} \, f_2' \, dz\Big|_{\partial\Omega_1} = [(udx + vdy)e^{-b(x,y)} \cos a(x,y) +$$

$$(vdx - udy)e^{-b(x,y)}\sin a \, (x,y)]_{\partial\Omega_1} = 0,$$

which leads to:

$$\sin a(x,y)\Big|_{\partial\Omega_1} = 0 \Rightarrow a(x,y)\Big|_{\partial\Omega_1} = n\pi$$

or

$$da(x,y)\Big|_{\partial\Omega_1} = a_x dx + b_y dy\Big|_{\partial\Omega_1} = 0$$

Using Cauchy-Riemann equations for

$$\alpha(z) = a(x,y) + ib(x,y)$$

we get:

$$b_y dx - b_x dy \Big|_{\partial\Omega_1} = \frac{\partial b}{\partial n}\Big|_{\partial\Omega_1} = 0.$$

The function $b(x,y)$ is a solution of the following problem:

$$\Delta b = 0 \quad \text{in} \quad \Omega$$

$$\frac{\partial b}{\partial n}\Big|_{\partial\Omega_1} = 0$$

$$b\Big|_{\partial\Omega_2} = 0$$

which has a unique solution

$$b(x,y) \equiv 0 \quad \text{in} \quad \Omega.$$

Since $\quad \alpha(z) = n\pi \quad$ in Ω, $\quad$ we get

$$f_2' = \pm f_1' \quad \text{in} \quad \Omega.$$

and

$$f_2 = \pm f_1$$

Another problem is:

$$\Delta U = 0 \quad (5) \text{ in } \Omega \quad \text{which is bounded by} \quad \partial\Omega_1 \cup \partial\Omega_2 \quad \text{with the}$$

boundary conditions:

$$\left.\frac{\partial U}{\partial n}\right|_{\partial\Omega_1} = 0 \tag{5a}$$

$$\left.|\nabla U|\right|_{\partial\Omega_2} = 1 \tag{5b}$$

$$U(x_o, y_o) = 0 \qquad\qquad (x_o, y_o) \in \Omega$$

Again we have the same theorem:

Theorem 2:

If $U = U_1$ is a solution of problem (5), then $U = \pm U_1$ are the only solutions of the problem.

The proof is the same as in theorem 1, but instead of conditons (2') and (3) we have:

$$\left.\text{Im } df\right|_{\partial\Omega_1} = \left.\text{Im } f'dz\right|_{\partial\Omega_1} = 0 \tag{6'}$$

if f_1' and f_2' are solutions

of the problem, then

$$f_2' = f_1' \, e^{i\alpha(z)}$$

and along $\partial\Omega_1$ the following holds:

$$\operatorname{Im} f_2' \, dz \Big|_{\partial\Omega_1} = \operatorname{Im} f_1' \, e^{i\alpha(z)} dz \Big|_{\partial\Omega_1} = ie^{-b} \sin a(udx + vdy) = 0$$

so,

$$\sin \, a(x,y) = 0 \quad \Rightarrow \quad a(x,y) = n\pi \, .$$

since

$$da(x,y) = a_x dx + a_y dy = b_y dx - b_x dy = \frac{\partial b}{\partial n}\Big|_{\partial\Omega_1} = 0$$

and

$$b(x,y)\Big|_{\partial\Omega_2} = 0$$

we get

$$b(x,y) \equiv 0$$

and

$$a(x,y) = n\pi \qquad \text{in } \Omega.$$

hence

$$f_2' = \pm f_1'$$

From the theorem above ., we conclude that the boundary problems (1), (5) have at most two solutions which differ only by sign. If the sign of the solution can be determined by the aerodynamic problem, the non-linear boundary value problem has a unique solution.

UNIQUENESS OF THE SOLUTION OF ELLIPTIC PROBLEMS
WITH MIXED BOUNDARY CONDITIONS

Sara Yaniv

Abstract

In the problem of designing shock-free supercritical air-
foils for the transonic small disturbance theory, certain
mixed boundary conditions arise.

In the following, the question of uniqueness of a
linear elliptic problem with the appropriate boundary
conditions is examined.

1. Introduction

The most important question in the theory of elliptic equations is the study of boundary value conditions for which the problem is well posed.

The most investigated problems are the first(Dirichlet),second (Neumann and third(mixed) boundary value problems. All these problems are special cases of the boundary conditions:

$$\gamma\frac{du}{dl} + \beta u = \phi \qquad \text{along the boundary},$$

see [2].

In what follows we prove that if the oblique derivative condition

$$\frac{du}{dl} = \frac{\partial u}{\partial n} + \alpha\frac{\partial u}{\partial s} = \phi$$

along the boundary is given,the Helmholtz equation is unique if $\alpha < 0$,which means that l is oriented towards the exterior of the region [2].

2. Boundary Value Problem

Consider the self-adjoint equation

(1) $\quad \Delta u = Pu \qquad$ in Ω

$\quad$ for $\quad P = P(x_1, x_2, \ldots, x_n) \geqslant 0$.

The boundary condition is:

(2) $\quad \dfrac{\partial u}{\partial n} + \alpha \dfrac{\partial u}{\partial s} = f \qquad$ along $\partial\Omega$,

$\quad$ where $\dfrac{\partial}{\partial n}$ is the outward normal derivative and $\dfrac{\partial}{\partial s}$ is the tangential derivative along the boundary. This condition is similar to the known mixed boundary condition for elliptic problem [1]:

(3) $\quad \dfrac{\partial u}{\partial n} + \alpha u = f$ along $\quad \partial\Omega$.

In the following theorem we prove that problem (1), (2) has a unique solution under the condition $\alpha < 0$, (the same holds for problem (1), (3)).

Theorem:

If $\quad \alpha(x_1, x_2 \ldots, x_n) < 0$ then problem (1), (2) has a unique solution.

Proof:

Using Green's identity we have:

$$(4) \quad \int_{\Omega} \left[\frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} + \ldots + \frac{\partial u}{\partial x_n} \frac{\partial v}{\partial x_n} + u \Delta v \right] dx_1 \ldots dx_n =$$

$$\int_{\partial \Omega} \left[\sum_{j=1}^{n} (-1)^{j-1} u \frac{\partial v}{\partial x_j} dx_1 \ldots dx_{j-1} \cdot dx_j \ldots dx_n \right]$$

Taking v a solution of equation (1) and $v = u$, then (4) reduces to:

$$(5) \quad \int_{\Omega} \left[\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2 + \ldots + \left(\frac{\partial u}{\partial x_n} \right)^2 + Pu^2 \right] dx = -\int_{\partial \Omega} u \frac{\partial u}{\partial n} dS.$$

Let us introduce the notation:

$$\| u \|^2 = \int_{\Omega} \left[\left(\frac{\partial u}{\partial x_1} \right)^2 + \ldots + \left(\frac{\partial u}{\partial x_n} \right)^2 + Pu^2 \right] dx \quad ,$$

since $P \geqslant 0$ this relation can be identified as a norm.

Consider two solutions u_1 and u_2 of the problem (1), (2) in Ω. Suppose $u_1(x_1, \ldots x_n)$ and $u_2(x_1, \ldots x_n)$ are smooth functions so that formula (5) can be applied to the difference $u = u_1 - u_2$.

Since u satisfies (1) in Ω we have

$$(6) \quad ||u||^2 = - \int_{\partial\Omega} u\frac{\partial u}{\partial n} \, dS,$$

and from equation (2)

$$\frac{\partial u}{\partial n} = -\alpha \, \frac{\partial u}{\partial s}$$

then

$$||u||^2 = \int_{\partial\Omega} \alpha u\frac{\partial u}{\partial s} \, dS = \frac{1}{2}\int_{\partial\Omega} \alpha \, \frac{\partial(u^2)}{\partial s} \, ds.$$

Using the mean-value theorem for the smooth function $\alpha(x_1,\ldots,x_n)$, we have:

$$||u||^2 - \frac{1}{2} \bar{\alpha} \int_{\partial\Omega} \frac{\partial(u^2)}{\partial s} \, dS = 0.$$

since $\quad \int_{\partial\Omega} \frac{\partial(u)^2}{\partial s} \, ds = m > 0$

then

$$||u||^2 - \frac{1}{2} \bar{\alpha} m = 0.$$

Assuming $\alpha < 0$ we obtain:

$$||u|| = 0$$

and

$$\frac{\partial (u^2)}{\partial s} = 0 \quad \text{on} \quad \partial\Omega.$$

Thus, uniqueness is assumed for problem (1), (2) provided $\alpha < 0$ and $P > 0$.

If $P(x_1, \ldots, x_n) = 0$

then

$$u_1 - u_2 = c.$$

which means that under the boundary condition (2), the solution of the Laplace equation

$$\Delta u = 0$$

can be determined up to an additive contant.

3.Numerical Examples

We solve Laplace equation in polar coordinates:

$$(6) \quad r^2 \phi_{rr} + r\phi_r + \phi_{\theta\theta} = f(r,\theta) \quad \text{in } \Omega$$

where

$$\Omega = \{(r,\theta)/r_o \leqslant r \leqslant R, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}\}$$

with the following boundary conditions:

$$(7a) \qquad \frac{\partial \phi}{\partial \theta}(r, \theta = 0) = g_1(r)$$

$$(7b) \qquad \phi(r, \theta = \frac{\pi}{2}) = g_2(r)$$

$$(7c) \qquad \phi(R, \theta) = h_1(\theta)$$

$$(7d) \qquad \frac{\partial \phi}{\partial r} \cos \theta - \frac{\partial \phi}{\partial \theta} \frac{\sin \theta}{r_o} \bigg|_{r = r_o} = h_2(\theta)$$

(figure 1).

Condition (7d) is of the form (2a) for which:

$$\alpha = - \frac{tg\, \theta}{r_o} < 0$$

for $\quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$

The numerical calculations are performed for the following examples:

I. The functions $f(r,\theta)$, $g_1(r)$, $g_2(r)$, $h_1(\theta)$ and $h_2(\theta)$ are chosen in such a way that the function

(8) $\phi(r,\theta) = r(\cos\theta + \sin\theta)$

is the solution of the problem.

The numerical solution converges to the analytic solution (8). Error estimates and numerical rate of convergence are given in table (I).

II. The above functions were chosen so that the function

$\phi(r,\theta) = 1/r$ is the solution of the problem. Results are given in table (II).

Remark :

Numerical experiments show that the condition $\alpha \leqslant 0$ is also necessary for uniqueness.

We tried to solve problems I and II for the half disc:

$$\Omega = \{(r,\theta) / r_0 \leqslant r \leqslant R, 0 \leqslant \theta \leqslant \pi\}$$

for which

$$\alpha = \frac{tg\,\theta}{r_0}$$

changes sign in the interval $0 \leqslant \theta \leqslant \pi$.

In this case the solution is not unique and we are unable to get a solution.

Numerical Results for the Laplace Equations

Let : $\Delta\theta = \dfrac{\pi}{2M}$

$$\Delta r = \frac{R - r_o}{N}$$

where $R = 1.6$

$r_o = 0.08$

$$E_{anal} = \left[\Sigma (\phi_{anal} - \phi_{num})^2 \, \Delta\theta\Delta r \right]^{1/2}$$

IDEG -   order of approximation of the normal derivative on the boundaries (first or - second order).

Table I: $\phi_{anal} = x + y = r(\cos\theta + \sin\theta)$

N	M	IDEG	EANAL
19	4	1	0.169
19	8	1	0.055
38	4	1	0.138
38	8	1	0.050
38	16	1	0.022
19	4	2	0.035
19	8	2	0.007
38	4	2	0.026
38	8	2	0.0064
38	16	2	0.0016

Table II: $\phi_{anal} = \frac{1}{r}$

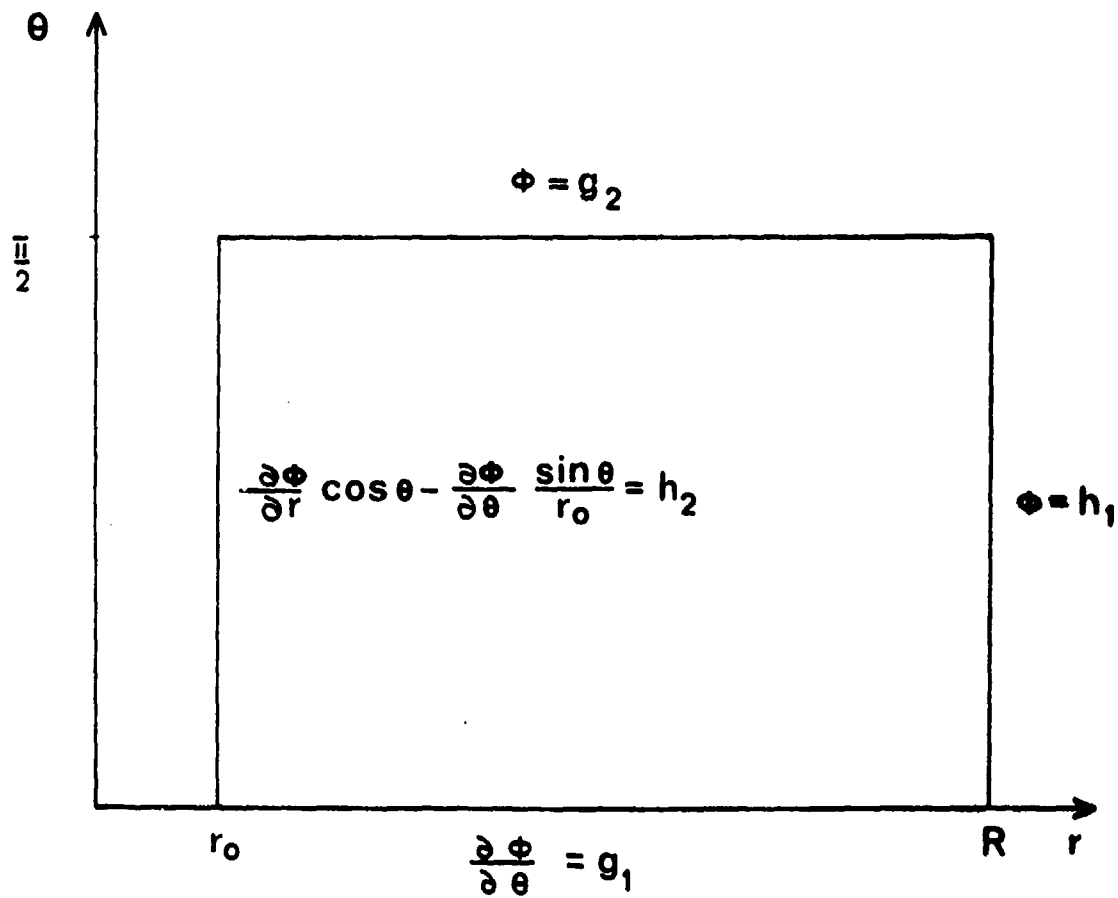| N	M	IDEG	Eanal
19	4	1	17.3
19	8	1	9.86
38	4	1	3.66
38	8	1	3.24
76	8	1	1.39
19	4	2	11.60
19	8	2	5.49
38	4	2	2.12
38	8	2	1.59
76	8	2	0.50

Figure 1.

References:

1. D.R. Garabedian: <u>Patial Differantial Equations</u>,
   John Wiley & Sons, Inc. 1964, Chapter 7.

2. Carlo Miranda . <u>Partial Differantial Equations Of Elliptic</u>
   <u>Type</u>, Springer-Verlag Berlin. Heildelberg. New York 1970.

NUMERICAL ASPECTS FOR A NEW DESIGN PROCEDURE OF

SUPERCRITICAL WINGS

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Abstract

The free sonic line method for two-dimensional transonic small disturbances equation in the subsonic region is described and results of numerical tests are given.

1. Introduction

A method for designing shock-free transonic configurations has been developed by Fung, Sobieczky and Seebass [1]. The flow about a given wing is solved for modified flow equations, using a fictitious gas in part of the field to guarantee elliptic conditions everywhere, except on a parabolic line, i.e., a sonic line. The velocity components along this sonic line serve as initial conditions for extension into the supersonic pocket terminating in a zero streamline curve, which modifies a part of the original airfoil to give a shock-free supersonic pocket.

Skipping the fictitious gas idea and calculations altogether, the proposal here is to prescribe the shape of the sonic line and get an elliptic problem in the subsonic region with new non-standard boundary conditions. The solution in the elliptic region and along the parabolic (sonic) line provides initial values for the hyperbolic pocket. The completion of the shockless flow field is carried out by continuation of the field into the hyperbolic pocket.

As a first step, we treat the transonic small disturbance equations, with boundary conditions given about a circular sonic line and a "circular" wing.

2. **Transonic small disturbance theory and formulation of the boundary value problem.**

The transonic small disturbance equations are derived by an asymptotic procedure applied to the exact equations of gas dynamics for the freestream mach number, $M\infty$, close to 1.

The small parameter of the expansion procedure is the airfoil thickness ratio δ, and the flow is represented as small pertrubation of a uniform stream. The limit process is associated with the approximation $\delta \rightarrow o$, $K, x, \bar{y}$ fixed where $\bar{y} = \delta^{1/3} y$ and $K = (1 - M\infty^2)/\delta^{2/3}$, a transonic similarity parameter [2].

The basic transonic equation is:

$$[K\phi_x - (\gamma + 1)\phi_x^2/2]_x + \phi_{\bar{y}\bar{y}} = 0.$$

or alternatively

$$(1) \qquad [K - (\gamma + 1)\phi_x]\phi_{xx} + \phi_{\bar{y}\bar{y}} = 0.$$

Equation (1) is hyperbolic for $\phi_x > K/(\gamma + 1)$, elliptic for $\phi_x < K/(\gamma + 1)$ and parabolic for $\phi_x = K/(\gamma + 1)$.

For flow around a wing, conditions of tangent flow have to be prescribed. For the first (and second) approximation the conditions in the plane of the wing, $\bar{y} = 0$, are:

$$\phi_{\bar{y}}(x,0) = \begin{cases} F'(x) & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

where the body shape is given by

$$y = \delta . F(x)$$

(the geometry of the problem and the boundary conditions are shown in figure 1).

In the design problem a wing for which shock-free flow occurs is to be constructed. In our procedure a symmetric sonic line Γ is chosen, resulting in the following boundary value-problem for the elliptic flow outside the sonic line:

(2) $\qquad [K - (\gamma + 1)\phi_x] \phi_{xx} + \phi_{\bar{y}\bar{y}} = 0$

(3) $\qquad \phi_{\bar{y}}(x,0) = \begin{cases} F'(x) & \xi < |x| < 1 \\ 0 & |x| > 1 \end{cases}$

(4) $\qquad \phi_x(x,y)|_\Gamma = K/(\gamma + 1)$ $\qquad\qquad$ (figure 2)

The mixed type condition (4) can be written as:

$$\phi_x\Big|_\Gamma = \frac{\partial \phi}{\partial n} \quad a(x) + \frac{\partial \phi}{\partial s} \quad b(x)\Big|_\Gamma = \frac{K}{(\gamma + 1)}$$

where $\frac{\partial \phi}{\partial n}$ is the derivative in the outward normal direction and $\frac{\partial \phi}{\partial s}$ is the tangential derivative. The boundary conditon (4) is the one we examined in [3]; uniqueness of the solution does not exist for every Γ.

For a numerical solution to the problem, boundary conditions for the far field are needed to replace the free stream condition $\bar{q} = U (1,0)$ at infinity.

Regarding equation (1) as an elliptic equation with a non-linear right-hand side we have:

$$L[\phi] = K\phi_{xx} + \phi_{\bar{y}\bar{y}} = [(\gamma + 1)/2] (u^2)_x$$

where $u = \phi_x$.

Applying Green's formula for L in Ω, assuming $\psi(\xi,\eta)$ continuous with continuous first derivatives, we have:

$$\iint_\Omega (\psi L[\phi] - \phi L[\psi])d\xi d\eta = \int_{\partial\Omega}\left(\psi\frac{\partial \phi}{\partial n} - \phi\frac{\partial \psi}{\partial n}\right) dl.$$

Taking $\psi(\xi,\eta,x,\bar{y})$ as the fundamental solution for the whole plane;

$$L_{\xi,\eta}[\psi] = \delta(x-\xi)\,\delta(\bar{y}-\eta)$$

$$\psi = S(\xi-x,\eta-\ddot{y})$$

and using a similar procedure as in [2], for the problem (2), (3) and (4) but omitting the possibility of shocks, we obtain:

$$\phi(x,\bar{y}) = \frac{\gamma+1}{2} \iint u^2(\psi_\xi + \psi^*_\xi)d\xi d\eta +$$

$$\int_{\partial\Omega}\left[\frac{\partial\phi}{\partial n}(\psi + \psi^*) - \phi\left(\frac{\partial\psi}{\partial n} + \frac{\partial\psi^*}{\partial n}\right)\right]dl.$$

where $\quad \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5.$ (Figure 3).

Along $\qquad \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$

$$\frac{\partial \psi}{\partial n} + \frac{\partial \psi^*}{\partial n} = 0$$

and along Γ_5:

$$-\int_{\Gamma_s} \phi \left(\frac{\partial \psi}{\partial n} + \frac{\partial \psi^*}{\partial n} \right) dl \approx \frac{2x}{x^2 + K\bar{y}^2} \cdot \frac{1}{2\pi \sqrt{K}} \int_{-\bar{\xi}}^{\bar{\xi}} \phi(\xi, n(\xi)) d\xi$$

where $\qquad n = n(\xi)$ describes the sonic line.

substituting the conditions:

$$\left. \frac{\partial \phi}{\partial n} \right|_{\Gamma_3 \cup \Gamma_4} = F'(\xi),$$

$$\left. \frac{\partial \phi}{\partial n} \right|_{\Gamma_s} = \left(\frac{\partial \phi}{\partial n}, -\frac{\partial \phi}{\partial \xi} \right) = \left(\frac{\partial \phi}{\partial n}, -K/(\gamma+1) \right)$$

and assuming, Γ_5 is a symmetric function of x, we get:

$$\int_{\Gamma_s} \frac{\partial \phi}{\partial \xi} (\psi + \psi^*) dn = 0$$

and
$$\int_{\partial\Omega} \frac{\partial\phi}{\partial\Omega}(\psi+\psi^*)dl = \int_{\Gamma_3 \cup \Gamma_4} F'(\xi)\ (\psi + \psi^*)dl +$$

$$\int_{\Gamma_s} \frac{\partial\phi}{\partial\eta}\ (\psi + \psi^*)dl$$

Let us define a function $G(\xi)$ by:

$$\frac{\partial\phi}{\partial\eta}\bigg|_{\Gamma_s} = \frac{\partial\phi}{\partial\eta}\ (\xi,\eta(\xi)) = G'(\xi) \qquad |\xi| < \bar{\xi}\ ,$$

since
$$\eta(\bar{\xi}) = \eta(-\bar{\xi}) = 0$$

and assuming continous velocities at the points $(-\bar{\xi},0),\ (\bar{\xi},0)$ where the sonic line touches the airfoil, we have:

$$F'(-\bar{\xi}) = G'(-\bar{\xi})$$
$$F'(\bar{\xi}) = G'(\bar{\xi}),$$

hence the function

$$H'(\xi) = \begin{cases} F'(\xi) & \bar{\xi} \leqslant |\xi| \leqslant 1 \\ G'(\xi) & |\xi| \leqslant \bar{\xi} \end{cases}$$

is a continuous function.

Using integration by parts and the fact that the wing is a closed curve, which means

$$F(\pm 1) = 0$$

we get:

$$\int_{\partial\Omega} \frac{\partial\phi}{\partial n}(\psi+\psi^*)dl = -\int_{-1}^{1} H(\xi)(\psi+\psi^*)_\xi d\xi$$

$$\approx \frac{1}{2\pi\sqrt{K}} \cdot \frac{2x}{x^2+K\bar{y}^2} \int_{-1}^{1} H(\xi)d\xi.$$

Hence, the potential in the far field is similar to the doublet for a closed body:

$$(5) \qquad \phi(x,\bar{y}) \approx \frac{1}{2\pi\sqrt{K}} \frac{Dx}{x^2+K\bar{y}^2} + \dots$$

where

$$(6) \qquad D = (\gamma+1)\iint_\Omega u^2 d\xi dy + 2\int_{-\bar{\xi}}^{\bar{\xi}}\phi(\xi,n(\xi))d\xi +$$

$$2\int_{-1}^{-\bar{\xi}} F(\xi)d\xi + 2\int_{\bar{\xi}}^{1} F(\xi)d\xi.$$

In the numerical procedure, D has to be calculated as one of the unknowns of the problem.

3. Circular sonic line

The transonic equation for the perturbation potential is

$$(K - (\gamma+1)\phi_x)\phi_{xx} + \phi_{\bar{y}\bar{y}} = 0.$$

Substituting

$$\bar{x} = \frac{x}{\sqrt{K}}$$

we obtain the equation:

$$(7) \qquad \phi_{\bar{x}\bar{x}} + \phi_{\bar{y}\bar{y}} = \frac{\gamma+1}{K\sqrt{K}} \phi_{\bar{x}} \phi_{\bar{x}\bar{x}}.$$

Equation (6) is solved for the elliptic part of the problem by assuming a given circular sonic line:

$$\Gamma_s: \quad \bar{x}^2 + \bar{y}^2 = r_o^2 < 1.$$

In polar coordinates:

$$\bar{x} = r \cos \theta$$
$$\bar{y} = r \sin \theta,$$

equation (6) becomes:

$$(8) \qquad \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2} \phi_{\theta\theta} = \frac{\gamma+1}{K\sqrt{K}}\Big[\phi_{rr}\cos^2\theta -$$

$$\phi_{r\theta} \frac{\sin 2\theta}{r} + \phi_{\theta\theta} \frac{\sin^2\theta}{r^2} + \phi_r\frac{\sin^2\theta}{r} +$$

$$\left. \phi_\theta \frac{\sin 2\theta}{r^2} \right] \cdot \left[\phi_r \cos\theta - \phi_\theta \frac{\sin\theta}{r} \right]$$

in the rectangle $\Omega' = \{(r,\theta)/r_0 \leqslant r \leqslant R_\infty, \ 0 \leqslant \theta \leqslant \pi \}$,
with the following boundary conditions:

(9a) $\qquad \phi_\theta(r, \theta=0) = \begin{cases} F'(r) \ r & r_0 \leqslant r \leqslant 1 \\ 0 & 1 < r < R_\infty \end{cases}$

(9b) $\qquad \phi_\theta(r, \theta=\pi) = \begin{cases} -r F'(-r) & r_0 \leqslant r \leqslant 1 \\ 0 & 1 < r < R_\infty \end{cases}$.

along the sonic line $r = r_0$:

(9c) $\qquad \phi_x = \phi_r \cos\theta - \phi_\theta \left. \frac{\sin\theta}{r} \right|_{r=r_0} = \frac{K\sqrt{K}}{\gamma+1}$

and the far field $r = R_\infty$

(9d) $\qquad \phi(R_\infty, \theta) = \dfrac{1}{2\pi\sqrt{K}} \dfrac{Dx}{x^2 + \bar{y}^2} = \dfrac{1}{2\pi K} \dfrac{D\bar{x}}{\bar{x}^2 + \bar{y}^2} = \dfrac{D}{2\pi K} \left. \dfrac{\cos\theta}{r} \right|_{r=R_\infty}$

where D is defined by (6).

Problem (8), (9) is not well posed, since the boundary
condition (9c):

$$\phi_r \cos\theta - \phi_\theta \frac{\sin\theta}{r} \bigg|_{r=r_0} = \frac{K\sqrt{K}}{\gamma+1} = 1/\alpha \qquad ,$$

for $\quad 0 \leqslant \theta \leqslant \pi$, does not satisfy the uniqueness condition:

$\dfrac{tg\theta}{r} > 0$ [3],

An additional requirement on the solution is needed to prescribe a well posed problem. Consider $\quad\quad$ an odd function of x , satisfying:

$$\phi(\bar{x},\bar{y}) = - \phi(-\bar{x},\bar{y}).$$

It is obvious that such a function is a solution.

The new probelm is: $\quad\quad\quad$ find the solution of equation (8) satisfying the following boundary conditions:

(10a) $\quad\quad \phi_\theta(r,\theta=0)\begin{cases} r\ F'(r) & r_o \leqslant r \leqslant 1 \\ 0 & 1 < r < R_\infty \end{cases}$

(10b) $\quad\quad \phi(r,\theta=\dfrac{\pi}{2}) = 0$

(10c) $\quad\quad \phi_r\cdot\cos\theta - \phi_\theta\dfrac{\sin\theta}{r}\bigg|_{r=r_o} =$

$\quad\quad\quad\quad\quad\quad\quad 0 \leqslant \theta \leqslant \pi/2$

and

(10d) $\quad\quad \phi(R_\infty,\theta) = \dfrac{D}{2\pi K}\cdot\dfrac{\cos\theta}{r}\bigg|_{\substack{r = R_\infty \\ 0 \leqslant \theta \leqslant \pi/2}}$

Numerical procedure

The non-linear problem is tested for some boundary conditions for which the analytic solution is known. Therefore we take a non-homogeneous equation which is solved by the following iterations:

$$(11) \qquad \left(1 - \alpha \phi_{\bar{x}}^{(n-1)}\right) \phi_{\bar{x}\bar{x}}^{(n)} + \phi_{\bar{y}\bar{y}}^{(n)} = f(\bar{x}, \bar{y})$$

Equation (11) is transformed into polar coordinates with the following boundary conditions:

$$\phi_x(r_0, \theta) = \phi_r \cos\theta - \phi_\theta \frac{\sin\theta}{r_0} = g_1(\theta), \quad 0 < \theta < \frac{\pi}{2}$$

$$\phi(R_\infty, \theta) = g_2(\theta) \qquad\qquad , \quad 0 < \theta < \frac{\pi}{2}$$

$$\phi_\theta(r, \theta=0) = g_3(r) \qquad\qquad , \quad r_0 \leqslant r \leqslant R_\infty$$

$$\phi(r, \theta = \frac{\pi}{2}) = g_4(r) \qquad\qquad , \quad r_0 \leqslant r \leqslant R_\infty$$

The domain $\Omega' = \{(r, \theta), \quad r_c \leqslant r \leqslant R_\infty \quad , \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}\}$
is devided into rectangles of size $\Delta\theta . \Delta r$

where $\Delta\theta = \frac{\pi}{2M}$ and $\Delta r = \frac{R_\infty - r_0}{N}$

For an interior point $((i,j), i = 2, \ldots, M), j = 2, \ldots, N)$
central differences are used in the equation.

On $\theta = 0$, where ϕ_θ is given as a boundary condition, we use either a first or a second order approximation.

On $r = r_o$, where $\phi_x = \phi_r \cos\theta - \phi_\theta \dfrac{\sin\theta}{r_o}$ is given, a central approximation for ϕ_θ is taken and ϕ_r is either a first or second order approximation, according to the order chosen for the boundary $\theta = 0$.

The schemes described before lead to a matrix which is tri-diagonal in blocks.

If first order approximations are used on $r = r_o$ then the size of all the blocks are the same: $(M+1)\times(M+1)$ and contain the coefficients of the schemes belonging to one row in the mesh.

If second order approximations are used on $r = r_o$ then each point on $r = r_o$ is connected to the rows $r_o + \Delta r$ and $r_o + 2\Delta r$. Therefore the order of one diagonal block will be $2(M+1) \times 2(M+1)$ and the size of the others is $(M+1) \times (M+1)$ only.

The system of equations is solved by a direct method of elimination by blocks which needs storage of twice the size of the greatest block .

The numerical procedure is stopped when

$$\left(\Sigma \left(\phi^{(n)} - \phi^{(n-1)} \right)^2 \right)^{1/2} < \epsilon .$$

Numerical results:

Let $\qquad \Delta\theta = \dfrac{\pi}{2M}$

$$\Delta r = \frac{R_\infty - r_0}{N}$$

where $\quad R \approx 1.6$

$\qquad r_0 \approx 0.08$

$$Eanal = \left[\Sigma(\phi_{anal} - \phi_{num})^2 \; \Delta\theta \; \Delta r \right]^{1/2}$$

IDEG - order of approximation on the boundaries
    (first of second order).

The numerical procedure is tried for the functions $\phi = \theta$ and $\phi = r$ for which the schemes are exact.
The process converges for every initial guess and the error decreases to zero for every size of mesh. If α increases the procedure is slower, i.e. more iterations are needed to achieve the same error.

For $\phi = x + y$ the errors are illustrated in table I and for $\phi = r^{1/2} \cos(\theta/2)$ in table II. In both cases the results do not depend on the initial guess. In the last example oscillations are observed on $r = r_0$ but they decrease when finer meshes are used and vanish for larger values of r_0. In order to achieve an elliptic problem for the non-linear equation (11) α is limited by 1 in the first example and $\alpha \leq 2\sqrt{r_0}$ (≈ 0.57) in the second example.

Table I: $\phi = x+y = r(\cos\theta + \sin\theta)$

N	M	IDEG	Eanal	α
19	4	1	0.170	
19	8	1	0.056	
38	4	1	0.138	
38	8	1	0.050	
				<u>0.01</u>
19	4	2	0.035	
19	8	2	0.007	
19	16	2	0.0017	
38	4	2	0.026	
38	8	2	0.006	
19	8	2	0.012	
19	16	2	0.003	<u>0.7</u>
38	8	2	0.011	

Table II: $\phi = r^{1/2} \cos(\theta/2)$

N	M	IDEG	E anal	α
19	4	1	0.107	
19	8	1	0.045	
38	8	1	0.019	
76	8	1	0.014	
38	16	1	0.010	
- - - - - - - - - - - - - - - - - -				0.1
19	4	2	0.027	
19	8	2	0.012	
38	8	2	0.0026	
76	8	2	0.0007	
38	16	2	0.0002	
- - - - - - - - - - - - - - - - - -				
19	8	2	0.0051	
38	8	2	0.00076	0.57
76	8	2	0.00015	
38	16	2	0.00061	

References

1. K-Y. Fung, H. Sobieczky and T. Seebass; Numerical Aspects of the Design of Shock-Free Wings. AIAA 12th Fluid and Plazma Dynamics Conference. July 23-25, 1979, Williamsburg, Virginia.

2. E.M. Murman and J.D. Cole; Calculation of Plane Steady Transonic Flows. AIAA Journal 1971, Vol. 9, No. 1. p. 114-121.

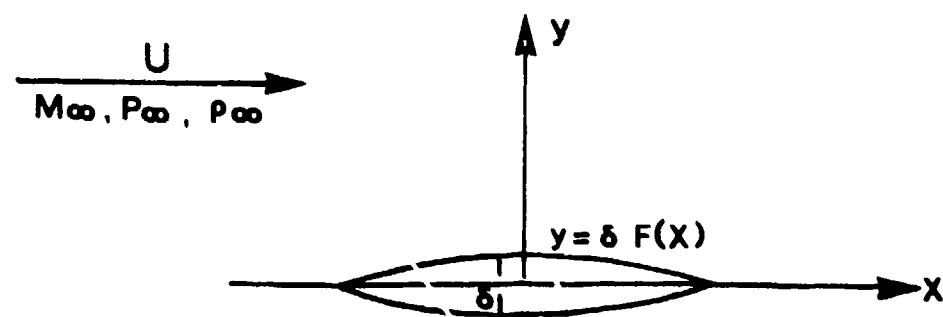3. S. Yaniv; Uniqueness of the Solution of Elliptic Problems with Special Mixed Boundary Conditions (this report).
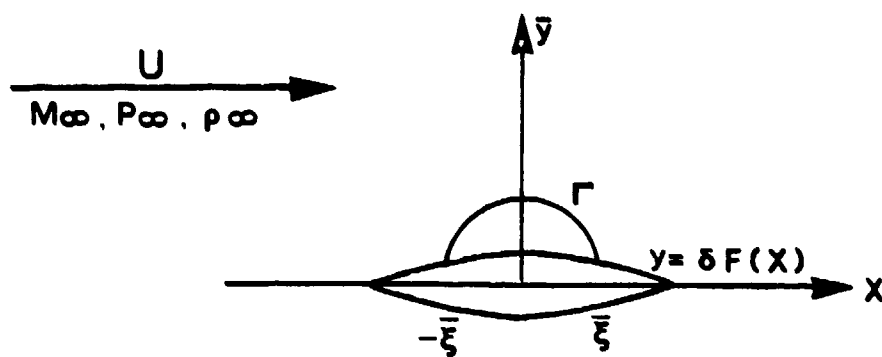
Figure 1: Transomic flow past a slender airfoil.
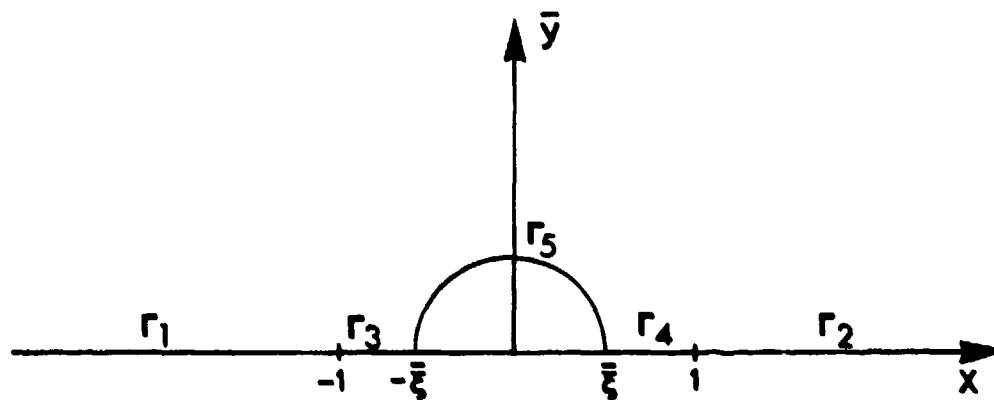
Figure 2.

Figure 3.

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